Joint Estimation of Default and Recovery Risk: A Simulation Study[†]

Jens Henrik Eggert Christensen
Dept. of Finance, Copenhagen Business School,
Solbjerg Plads 3, 2000 Frederiksberg, Denmark

Abstract

A joint modeling and estimation of recovery risk, default intensity risk, and interest rate risk has so far been a neglected issue in the empirical credit risk literature. This paper presents a framework in which all three types of risk can be modeled. It is argued that under the assumption of recovery of face value it is theoretically possible to jointly estimate both recovery and default intensity risk. Given simulated data, a model containing all three types of risk is estimated; the results show that default and recovery risk can in fact be separated.

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1 Introduction

In the empirical credit risk literature much attention has been paid to modeling and explaining the credit spread of corporate bonds and credit default swaps, Duffee (1999) and Driessen (2005) are excellent examples as far as corporate bonds are concerned. However, given the continued growth of the credit derivatives markets, in particular the remarkable growth of the default swap market,[1] and the constant advent of new products, most recently default digital swaps and recovery lock products (see Liu, Naldi, and Pedersen (2005) for details), it becomes evermore imperative to decompose the credit risk into a recovery risk component and a default risk component. Thorough knowledge of the dynamic structure of the market-implied risk premia attached to default and recovery risk could play a key role in portfolio risk management as well as for calculating economic capital requirements. The ability to contrast this knowledge with the vast literature on actual default probabilities[2] and observed recovery rates on defaulted debt claims would give a better overall understanding of the basic nature of credit risk and its components.

Bakshi, Madan, and Zhang (2004) were the first to attempt such a decomposition of default and recovery risk. Unfortunately, their model suffers from the fact that there is only one factor, the interest rate, to explain both interest rate, default, and recovery risk. More recently, Pan and Singleton (2005) estimate both the market-implied default risk and the market-implied recovery rate from observed sovereign credit default swap spreads for three major emerging market economies (Mexico, Turkey, and Russia). However, their model contains a constant recovery rate, therefore no knowledge about the dynamics of market-implied recovery rates can be derived. Neither of these papers allows for the joint estimation of the market-implied risk premia on default and recovery risk; this paper addresses this issue.

The framework used in this paper is the first to combine the following notable observations from the credit risk literature on corporate bonds:

(*i*). There is a growing body of empirical work on recovery rates indicating that stochastic modeling of the recovery rate is essential.

(*ii*). An elaborate modeling of the default intensity λ_t is necessary to match observed yield spreads.

(*iii*). Guha (2002) finds in a study of defaulted bonds that at, and in a period after, default bonds of equal seniority trade at identical prices, independent of their remaining time to maturity or coupon size. This recovery pattern is most closely modeled through the assumption of recovery of face value (RFV).

[1] According to ISDA's Year-End 2004 Market Survey, the outstanding notional of credit default swaps alone had reached \$8.42 trillion by the end of 2004, up 55% during the last 6 months of that year.

[2] For an example see reports from ratings agencies like Moody's and Standard and Poor's.

Combining these three features in a reduced-form setting, the price of a defaultable coupon bond with maturity in T years and N outstanding coupon payments C at dates $t_1, \ldots, t_N$ is given by[3]

$$\begin{aligned} V^{C,RFV}(T) &= E_0^Q\Big[\exp\Big(-\int_0^T (r(X_u)+\lambda(X_u))du\Big)\Big] \\ &+ \sum_{i=1}^{N} C E_0^Q\Big[\exp\Big(-\int_0^{t_i} (r(X_u)+\lambda(X_u))du\Big)\Big] \\ &+ \int_0^T E_0^Q\Big[\pi(X_s)\lambda(X_s)\exp\Big(-\int_0^s (r(X_u)+\lambda(X_u))du\Big)\Big]ds, \end{aligned}$$

where $r(X_t)$ is the risk-free instantaneous interest rate, $\lambda(X_t)$ is the default intensity, $\pi(X_t)$ is the stochastic recovery rate, and X_t is a multi-dimensional vector of state variables.

This paper exploits the asymmetry between the default intensity $\lambda(X_t)$ and the recovery rate $\pi(X_t)$ in the price of a defaultable bond under the RFV assumption in order to investigate the possibility of decomposing corporate bond spreads into default and recovery risk. To do this, one must first create a modeling framework that allows expectations of the form

$$E_0^Q\Big[\pi(X_t)\lambda(X_t)\exp\Big(-\int_0^t (r(X_u)+\lambda(X_u))du\Big)\Big]$$

to be easily calculated.

A widely popular class of models for bond pricing is that of affine term structure models, see Duffie and Kan (1996) and Duffie, Pan, and Singleton (2000). The advantage of these models is that they provide closed-form solutions up to the solution of a set of ODEs, and the bond price itself is a sum of exponential-affine functions of the state variables. For the purposes of this paper, the sole relevant limitation is that they only allow for affine functions of the state variables in the pricing kernel. This implies that in an affine setting, with a full-scale modeling of the default intensity $\lambda(X_t)$, the recovery rate $\pi(X_t)$ has to be modeled either as a constant or as an exponential-affine function of the state variable $\pi(X_t) = \pi_0 + \pi_1 e^{-h'X_t}$ (see Bakshi, Madan, and Zhang (2004) for an application of this).

Now, if one decides to model both the default intensity $\lambda(X_t)$ and the recovery rate $\pi(X_t)$ as affine functions of the state variable X_t, the product $\pi(X_t)\lambda(X_t)$ will be a quadratic form of X_t. The class of quadratic term structure models has the same advantage as the affine class, they provide closed-form solutions up to the solution of a set of ODEs, and they give bond prices as a sum of exponential-quadratic functions of the state variables. Filipović (2002) proves that the quadratic class is the maximal size of a polynomium of the state variables in the pricing kernel to possess this exponential-polynomial time-separable feature. Beyond second degree polynomiums, bond price calculations involve solving PDEs directly, which

[3]This formula does not take into account any accrued interest at default.

is prohibitively time-consuming in any multi-dimensional setting; hence, the current paper's focus on the class of quadratic term structure models. Additionally, the quadratic class is sufficient to significantly expand the recovery modeling options when default and recovery risk are jointly modeled.[4]

Several approaches have been applied in the literature on the estimation of quadratic models, but they have all focused on estimating the treasury yield curve and comparing the results to the performance of the affine term structure models. Ahn, Dittmar, and Gallant (2002) use the EMM procedure of Gallant and Tauchen (1996) to show that quadratic models better fit the conditional volatility of bond yields than affine models of a similar dimension - the latter are actually very poor in catching these effects. In addition, they discuss the issue of identification in the multi-dimensional quadratic model and the handling of risk premia. Leippold and Wu (2003) identify a set of stylized features of observed US interest rates over a 15-year period, specifically the hump-shaped conditional volatility of yields. They prove that it takes at least a two-dimensional quadratic model to match the documented features, and they use a GMM estimation to fit a parsimonious two-factor quadratic model to the data that possesses the desired qualities. Kim (2004) estimates quadratic models on 25 years of US interest rate data using the extended Kalman filter method. He linearizes the quadratic measurement equation by augmenting the state space with the quadratic terms of the state variables and demonstrates how to perform the quasi maximum likelihood estimation of the Kalman filter in this case. He then investigates the relationship between the risk premia flexibility and the quadratic models' ability to describe the stochastic volatility of yields. His study shows that in the quadratic class there is no tension between the risk premia modeling and the volatility modeling unlike what holds in the affine class (for a discussion of this issue see Dai and Singleton (2002)).

It should be noted that a potential limitation of working within the quadratic class is that the state variables are restricted to the affine class of processes with a non-stochastic volatility matrix (see Leippold and Wu (2002)). The first time the quadratic class of term structure models was treated with the same level of generality as Duffie, Pan, and Singleton (2000) had treated the affine class, was in Leippold and Wu (2002). However, Leippold and Wu (2002) only provided the set of ODEs for the standard quadratic pricing kernel

$$E_0^Q\Big[\exp\Big(-\int_0^T (X_u' A_r X_u + B_r' X_u + C_r)du\Big)\exp(X_T'\overline{A}X_T + \overline{B}'X_T + \overline{C})\Big].$$

Using the same approach as in Duffie, Pan, and Singleton (2000), this paper extends the

[4] For example, given an affine function for the default intensity $\lambda(X_t) = \lambda_0 + \lambda_1' X_t$, the most general form of the recovery rate that keeps the model within the quadratic class is of the form

$$\pi(X_t) = \pi_0 + \pi_1' X_t + \pi_2 \exp(X_t'\overline{A}_\pi X_t + \overline{B}_\pi' X_t) + \pi_3' X_t \exp(X_t'\widetilde{A}_\pi X_t + \widetilde{B}_\pi' X_t). \tag{1}$$

result of Leippold and Wu (2002) to encompass the extended quadratic pricing kernel

$$E_0^Q\Big[\exp\Big(-\int_0^T (X_u' A_r X_u + B_r' X_u + C_r)du\Big)\exp(X_T'\overline{A}X_T + \overline{B}'X_T + \overline{C})[X_T'\overline{D}X_T + \overline{E}'X_T + \overline{F}]\Big].$$

Now that a modeling framework has been established, a simple three factor model containing affine processes for the interest rate risk, the default intensity risk, and the recovery risk is applied and simulated. The subsequent estimation is performed in two steps. First, the path and parameters of the interest rate process are estimated from simulated default-free treasury bond yields. Then, given the result of the interest rate estimation, a joint estimation of the default intensity risk process and the recovery risk process is performed based on the simulated corporate bond yields. For both estimations the standard extended Kalman filter method is applied. The advantage of the Kalman filter approach is that it allows one to estimate by quasi maximum likelihood the most likely parameters and paths of the factors without making any unnecessary assumptions. Furthermore, unreported tests show that practically nothing will be gained by applying the state space augmentation proposed by Kim (2004).

The results of the estimation show that, with limited noise in the bond yields, it is possible to distinguish the default intensity risk and the recovery risk. However, despite the asymmetry implied by the RFV assumption, there is still, in general, a fundamental identification problem that needs to be addressed. In the future, new credit derivative products like default digital swaps may provide a way to bypass this problem.

The rest of this paper is structured as follows. Section 2 generalizes the multi-dimensional treatment of the standard quadratic model in Leippold and Wu(2002) to the extended version of the quadratic class. Section 3 describes the three-factor model, how it is simulated, and the specific choice of parameters. Section 4 discusses the Kalman filter estimation, its results, and the problem of separating the default and recovery risk components. Section 5 concludes the paper.

2 The differential equations of the extended quadratic model

This section lays out the mathematical results needed in order to calculate expectations of the type

$$E^Q\Big[\pi(X_t)\lambda(X_t)\exp\Big(-\int_0^t (r(X_u)+\lambda(X_u))du\Big)\Big|\mathcal{F}_0\Big],$$

where $\lambda(X_t)$ is affine and $\pi(X_t)$ has a functional form similar to Equation (1) in Footnote 4.

The state variable processes in the quadratic class of term structure models are restricted to the affine class with non-stochastic volatility matrices (see Leippold and Wu (2002)). Given a fixed propability space $(\Omega, \mathcal{F}, P)$, where the filtration $(\mathcal{F}_t) = \{\mathcal{F}_t : t \geq 0\}$ satisfies the usual conditions, see Williams (1997), the state variable X_t is therefore restricted to be an N-

dimensional Markov process defined on $\mathbf{R}^N$ that solves a system of stochastic differential equations of the following form

$$dX_t = [\mu^0(t) + \mu^1(t)X_t]dt + \Sigma(t)dW_t. \tag{2}$$

Here, W is a standard Brownian motion in $\mathbf{R}^d$, whose information is contained in the filtration $(\mathcal{F}_t)$, and $\mu^0 : [0,T] \to \mathbf{R}^N$, $\mu^1 : [0,T] \to \mathbf{R}^{N\times N}$, and $\Sigma : [0,T] \to \mathbf{R}^{N\times d}$ are bounded, continuous functions.[5]

The term 'quadratic' refers to the chosen function for the instantaneous short rate process which is assumed to be a quadratic form in the state variables

$$r(X_t, t) = X_t' A^r(t) X_t + B^r(t)' X_t + C^r(t).$$

If $A^r(t)$ is a semidefinite matrix and the following equality holds

$$C^r(t) - \frac{1}{4} B^r(t)' A^r(t)^{-1} B^r(t) \geq 0, \quad \forall t \geq 0,$$

then the short rate process will be strictly positive a.s.

The following proposition is an extension of the result in Leippold and Wu (2002) for the quadratic class of term structure models

Proposition 1:

Let X_t be a stochastic process of the type described by (2), then the expectation

$$\begin{aligned} G(X_t, t, T) &= E\big[e^{-\int_t^T r(X_s,s)ds} e^{X_T'\overline{A}X_T + \overline{B}'X_T + \overline{C}} [X_T'\overline{D}X_T + \overline{E}'X_T + \overline{F}] | \mathcal{F}_t\big], \\ & \quad \overline{A}, \overline{D} \in \mathbf{R}^{N\times N}, \quad \overline{B}, \overline{E} \in \mathbf{R}^N, \quad \overline{C}, \overline{F} \in \mathbf{R}, \end{aligned}$$

where $r(X_t, t) = X_t' A^r(t) X_t + B^r(t)' X_t + C^r(t)$ and $A^r : [0,T] \to \mathbf{R}^{N\times N}$, $B^r : [0,T] \to \mathbf{R}^N$ and $C^r : [0,T] \to \mathbf{R}$ are all bounded and measurable functions, has the solution

$$G(X_t, t, T) = e^{X_t' A(t,T) X_t + B(t,T)' X_t + C(t,T)} [X_t' D(t,T) X_t + E(t,T)' X_t + F(t,T)],$$

if the following conditions are met

(i). There is a unique solution to the stochastic differential equation (2) of X_t for $0 \leq t \leq T$.

(ii). There exist functions $A(t,T)$, $B(t,T)$, $C(t,T)$, $D(t,T)$, $E(t,T)$, and $F(t,T)$ which are the unique solutions to the following system of ordinary first order differential equations

$$\frac{dA(t,T)}{dt} = A^r(t) - A(t,T)\mu^1(t) - \mu^1(t)' A(t,T) - 2A(t,T)\Sigma(t)\Sigma(t)' A(t,T), \quad A(T,T) = \overline{A},$$

[5]Stationarity of the state variables is ensured, if all the eigenvalues of $\mu^1(t)$ are negative (if complex, the real component should be negative), see Ahn et al. (2002).

$$\frac{dB(t,T)}{dt} = B^r(t) - 2A(t,T)\mu^0(t) - \mu^1(t)'B(t,T) - 2A(t,T)\Sigma(t)\Sigma(t)'B(t,T), \quad B(T,T) = \overline{B},$$

$$\frac{dC(t,T)}{dt} = C^r(t) - B(t,T)'\mu^0(t) - tr(A(t,T)\Sigma(t)\Sigma(t)') - \frac{1}{2}B(t,T)'\Sigma(t)\Sigma(t)'B(t,T), \quad C(T,T) = \overline{C},$$

$$\frac{dD(t,T)}{dt} = -D(t,T)\mu^1(t) - \mu^1(t)'D(t,T) - 4A(t,T)\Sigma(t)\Sigma(t)'D(t,T), \quad D(T,T) = \overline{D},$$

$$\frac{dE(t,T)}{dt} = -2D(t,T)\mu^0(t) - 2D(t,T)\Sigma(t)\Sigma(t)'B(t,T) - \mu^1(t)'E(t,T) - 2A(t,T)\Sigma(t)\Sigma(t)'E(t,T),$$
$$E(T,T) = \overline{E},$$

$$\frac{dF(t,T)}{dt} = -E(t,T)'\mu^0(t) - tr(D(t,T)\Sigma(t)\Sigma(t)') - B(t,T)'\Sigma(t)\Sigma(t)'E(t,T), \quad F(T,T) = \overline{F}.$$

(iii). The following two technical conditions are met

(a) $E\Big[\big(\int_0^T \eta_t'\eta_t dt\big)^{1/2}\Big] < \infty$ for
$\eta_t = \big(\Phi_t[2X_t'A(t,T) + B(t,T)'] + \Psi_t[2X_t'D(t,T) + E(t,T)']\big)\Sigma(t)$.

(b) $E\big[|\Phi_T|\big] < \infty$.

where $\Phi_t = e^{\int_0^t r(X_s,s)ds} e^{X_t'A(t,T)X_t + B(t,T)'X_t + C(t,T)}[X_t'D(t,T)X_t + E(t,T)'X_t + F(t,T)]$ and $\Psi_t = e^{\int_0^t r(X_s,s)ds} e^{X_t'A(t,T)X_t + B(t,T)'X_t + C(t,T)}$ for all $0 \leq t \leq T$.

Proof: See Appendix A.

The proof merely consists of an application of Ito's lemma to the extended exponential-quadratic function Φ_t. If $A(t,T), \ldots, F(t,T)$ are the solutions to the system of ODEs outlined in the proposition (and the additional technical conditions are satisfied), then Φ_t is a martingale, and the result follows.

The result developed above is the extension of Leippold and Wu (2002) for the quadratic term structure models, similar to the extension by Duffie, Pan, and Singleton (2000) of the result in Duffie and Kan (1996) for the affine term structure models. To calculate the expectation of the extended pricing kernel one must first solve the same set of ODEs as when calculating the expectation of the standard pricing kernel, in particular $A(t,T)$ and $B(t,T)$. Once this is done the additional set of ODEs must be solved to obtain the solutions of $D(t,T)$, $E(t,T)$, and $F(t,T)$.[6]

As for the three-factor model used later in this paper, the result in the proposition above shows that the relevant expectations needed to calculate corporate bond prices will take the

[6]The same two-stage separating principle holds when solving the ODEs for the extended pricing kernel in the affine term structure models.

form

$$E^Q\Big[\pi(X_t)\lambda(X_t)\exp\Big(-\int_0^t(r(X_u)+\lambda(X_u))du\Big)\Big|\mathcal{F}_0\Big] =$$
$$e^{B(t,T)'X_t+C(t,T)}[X_t'D(t,T)X_t+E(t,T)'X_t+F(t,T)],$$

where $B(t,T),\dots,F(t,T)$ are easily solved by Runge-Kutta methods.

3 A simple three-factor model of corporate bond pricing

In Section 3.1 it will be argued that Recovery of Face Value as a recovery assumption is not only important for the purpose of distinguishing default and recovery risk but is also the most appropriate assumption in order to theoretically replicate the legal claim of bond holders at default. In Section 3.2 a simple model is developed that contains interest rate, default, and recovery risk while allowing for a fast calculation of bond prices. The set of parameter values chosen for the simulation of the model is presented in Section 3.3. The actual simulation and estimation are described in Section 3.4 and Section 3.5, respectively. The results of the estimation are deferred to Section 4.

3.1 The recovery assumption and the price of a corporate coupon bond

Let r_t denote the risk-free instantaneous interest rate. In reduced-form credit risk modeling the default event of a firm is modeled as the first jump of a doubly-stochastic Poisson process, independent of the state variables, with arrival intensity λ_t. The default time is denoted by τ. Now, consider a defaultable corporate bond with maturity at T and coupon C paid semi-annually, i.e. there is a set of coupon dates equal to $t_1=\frac{1}{2},\dots,t_N=T$, where $N=2T$ and, by definition, $t_0=0$ is not a coupon date. The formula applied in order to calculate prices of such bonds depends critically on the recovery assumption. The difference between the various recovery assumptions lies in what the recovery rate π_t is a fraction of.

In the credit risk literature several recovery assumptions have appeared, most notably the Recovery of Market Value, the Recovery of Treasury, and the Recovery of Face Value.

The Recovery of Market Value (RMV) assumption is analyzed extensively in Duffie and Singleton (1999). In the setting of a reduced-form model where the jump of the Poisson point process that signals the default event is completely unpredictable, this recovery assumption simply says that at default the bond holder receives an amount equal to a fraction π_τ of the market value prevailing immediately before the default event - a state where everything is fine, as the default is unpredictable. However, it is not entirely clear what the real world equivalent is to the theoretical notion of 'the market value immediately before the default event'. Historically, bond prices of companies that have ultimately filed for bankruptcy have had a tendency to decline in the weeks or even months before the actual filing by a mixture of outright downward jumps and gradual declines due to the market's loss of faith.[7] Thus, if the market price one business day before the filing is used as a proxy for the theoretical notion, the recovery rate will be significantly overstated (if one associates a recovery rate to mean a recovered amount relative to the original principal).

As for the purpose of the current paper, the RMV assumption has an additional unattractive feature, which becomes evident when looking at the actual bond price formula derived by Duffie and Singleton (1999). Under the assumption that every coupon payment recovers the same fraction of their pre-default market value as the principal,[8] the formula for the price of a corporate bond is[9]

$$V^{C,RMV}(t,T) = E_t^Q[e^{-\int_t^T (r_u+(1-\pi_u)\lambda_u)du}] + \sum_{t_i>t}^{N} \frac{C}{2} E_t^Q[e^{-\int_t^{t_i} (r_u+(1-\pi_u)\lambda_u)du}].$$

Since the default intensity λ_t and the recovery rate π_t only appear as a joint product $(1-\pi_u)\lambda_u$, it is impossible to distinguish between default and recovery risk, only the total credit spread is observable. Thus, given the purpose of this paper the RMV assumption is not useful.

A second recovery assumption is the Recovery of Treasury (RT) assumption applied by Jarrow and Turnbull (1995). Here, the recovered amount is a fraction π_t of the value of an otherwise identical treasury bond, i.e. same time to maturity, coupon size, and payment dates.

The bond pricing formula for the corporate bond under the RT assumption takes the following form

$$\begin{aligned} V^{C,RT}(t,T) &= E_t^Q[\mathbf{1}_{\{\tau>T\}} e^{-\int_t^T r_u du}] + \sum_{t_i>t}^{N} E_t^Q[\frac{C}{2}\mathbf{1}_{\{\tau>t_i\}} e^{-\int_t^{t_i} r_u du}] \\ &+ E_t^Q\Big[\int_t^T \pi_s \mathbf{1}_{\{s<\tau\leq s+ds\}} e^{-\int_t^s r_u du}\Big(e^{-\int_s^T r_u du} + \sum_{t_i>s}^{N} \frac{C}{2} e^{-\int_s^{t_i} r_u du}\Big) ds\Big]. \end{aligned}$$

Using the well-known conditioning principle for Cox processes, see Lando (1998), this can be

[7] Lando (2004) p. 121 has an illustration of this for the Enron Corporation that filed for Chapter 11 on 2 December 2001.

[8] This must be the case in order for the concept of pre-default market value to refer to just one price.

[9] The advantage of the RMV assumption and the reason why it is widely used in the empirical credit risk literature is that all the mathematical tools traditionally applied within standard term structure theory for calculating bond prices can be immediately applied to calculate corporate bond prices.

written as

$$
\begin{aligned}
V^{C,RT}(t,T) &= E_t^Q[e^{-\int_t^T (r_u+\lambda_u)du}] + \sum_{t_i>t}^{N} \frac{C}{2} E_t^Q[e^{-\int_t^{t_i}(r_u+\lambda_u)du}] \\
&+ \int_t^T E_t^Q\left[\pi_s \lambda_s e^{-\int_t^s (r_u+\lambda_u)du}\left(E_s^Q[e^{-\int_s^T r_u du}] + \sum_{t_i>s}^{N} \frac{C}{2} E_s^Q[e^{-\int_s^{t_i} r_u du}]\right)\right] ds.
\end{aligned}
$$

From the formula it is evident that the RT assumption allows for a separation of default and recovery risk.

To the author's best knowledge, Bakshi, Madan, and Zhang (2004) is the only paper to date that has examined the appropriateness of the recovery assumption for the empirical separation of default and recovery risk. They compare the RT and RFV assumptions and find that the RT assumption performs better on their data of BBB-rated corporate bonds in out-of-sample tests and in terms of minimizing the pricing error. However, the interest rate is the only stochastic factor in their model and their sample only consists of data on 25 companies; so to draw firm conclusions based on their analysis is questionable. It should be noted, however, that they do find that the RFV assumption implies more stable estimates of expected recovery rates, and for that reason they argue that the RFV assumption might be preferred when writing recovery related contingent claims.

Under the Recovery of Face Value (RFV) assumption the bond holder immediately receives the following payoff in the event of a default before maturity

- A fraction π_τ of the face value of the bond.
- A fraction π_τ of the accrued interest since the last coupon date.

Guha (2002) studies the market prices at and around default of different corporate bonds issued by the same company and belonging to the same seniority class. The first part of his paper studies the legal details of standard bond indentures and the US bankruptcy code. Most bond indentures state that in the event of default, the principal, including any accrued interest, immediately becomes due. From an economic point of view, coupon payments, while a legal claim, can be considered to be no more than compensation for postponed consumption. The implication of this line of thinking is that defaulted bonds of the same seniority should trade at identical prices in the secondary market, independent of the coupon size or the remaining time to maturity. However, deviations from this pattern can be observed in the market for several reasons. In Chapter 11 proceedings[10] claims are grouped into different classes, mainly according to seniority. The bankruptcy code states that all claims in the

[10] In Guha (2002)'s sample of defaulted companies, 32 out of the 34 US domiciled companies ultimately filed for Chapter 11 after they had defaulted on their debt, so one can expect a large majority of US corporate bankruptcies to eventually go through this procedure.

same class must be treated in the same way.[11] From an ex ante point of view, at the time of default it can be difficult to envision how the dynamic structure of the negotiation game caused by the various voting rules in the code will play out, and to predict in which class a specific claim may be placed. This uncertainty can cause claims of equal seniority to trade at different prices at the time of default.

In his empirical analysis, Guha (2002) finds that in 80% of the cases, bonds of the same seniority trade within a band of $1 per $100 principal at the time of default, independent of coupon size and time to maturity, which is what would be expected under the RFV assumption. In addition, he also investigates whether the observed price patterns could be fitted by other recovery assumptions, in particular the RMV and RT assumptions discussed earlier. He does not find that any of these alternatives are superior to the RFV assumption.

Under the RFV assumption the basic pricing equation of the corporate coupon bond under consideration is given by

$$\begin{aligned} V^{C,RFV}(t,T) &= E_t^Q\Big[\mathbf{1}_{\{\tau>T\}}e^{-\int_t^T r_u du}\Big] + \sum_{t_i>t}^N E_t^Q\Big[\frac{C}{2}\mathbf{1}_{\{\tau>t_i\}}e^{-\int_t^{t_i} r_u du}\Big] \\ &+ E_t^Q\Big[\int_t^T \pi_s \mathbf{1}_{\{s<\tau\le s+ds\}}e^{-\int_t^s r_u du}ds\Big] \\ &+ \sum_{t_i>t}^N E_t^Q\Big[\int_{t_{i-1}}^{t_i} \pi_s \frac{C}{2}\frac{s-t_{i-1}}{t_i-t_{i-1}}\mathbf{1}_{\{s<\tau\le s+ds\}}e^{-\int_t^s r_u du}ds\Big]. \end{aligned}$$

The two terms in the first line equal the expected discounted value of principal and coupon payments, respectively, conditional upon survival. The remaining two terms refer to the discounted value of the recovered amount of the principal and the accrued interest at default, respectively.

Again, using the conditioning principle for Cox processes, this can be written as

$$\begin{aligned} V^{C,RFV}(t,T) &= E_t^Q[e^{-\int_t^T (r_u+\lambda_u)du}] + \sum_{t_i>t}^N \frac{C}{2}E_t^Q[e^{-\int_t^{t_i}(r_u+\lambda_u)du}] \\ &+ \int_t^T E_t^Q[\pi_s\lambda_s e^{-\int_t^s (r_u+\lambda_u)du}]ds \\ &+ \sum_{t_i>t}^N \int_{t_{i-1}}^{t_i} \frac{C}{2}\frac{s-t_{i-1}}{t_i-t_{i-1}}E_t^Q[\pi_s\lambda_s e^{-\int_t^s (r_u+\lambda_u)du}]ds. \end{aligned}$$

This formula shows that under the RFV assumption the recovery rate π_t and the default intensity λ_t appear asymmetrically. This allows for the possibility of estimating the properties of both processes from observed corporate bond prices. Combining this with the economic

[11] Of course unless the claimants have given their consent to be treated otherwise.

intuition and the empirical support in Guha (2002), this paper will adopt the Recovery of Face Value assumption.

3.2 The model

The mathematical result in Section 2, Proposition 1, provides a modeling framework in which all expectations in the RFV bond price formula are easily calculated. If the interest rate r_t is set equal to one of the state variables, and the default intensity λ_t and the recovery rate π_t are linearly correlated with r_t then, in order to remain within the quadratic class, the state variables must be confined to the class of affine processes with non-stochastic volatility matrix.

The risk-free instantaneous interest rate is thus assumed to be of the Vasicek type

$$dr_t = \kappa_r(\theta_r - r_t)dt + \sigma_r dW_t^r.$$

Besides the interest rate process, the state variables consist of two independent factors also of the Vasicek type

$$\begin{aligned} dX_t^\lambda &= \kappa_\lambda(\theta_\lambda - X_t^\lambda)dt + \sigma_\lambda dW_t^\lambda, \\ dX_t^\pi &= \kappa_\pi(\theta_\pi - X_t^\pi)dt + \sigma_\pi dW_t^\pi. \end{aligned}$$

The default intensity and the recovery rate are assumed to be affine functions of the factors

$$\begin{aligned} \lambda_t &= \lambda_0 + \lambda_r(r_t - \theta_r) + \lambda_1(X_t^\lambda - \theta_\lambda), \\ \pi_t &= \pi_0 + \pi_r(r_t - \theta_r) + \pi_1(X_t^\pi - \theta_\pi). \end{aligned}$$

The affine formulation is chosen because it is simple, allows for positive as well as negative correlation between the interest rate on one side and the default intensity and the recovery rate on the other, and keeps the model within the quadratic class as far as bond pricing is concerned.

The assumption of Gaussian state variables has both advantages and disadvantages. As for the estimation to be performed, the Gaussian state variables imply that the state equation of the Kalman filter becomes Gaussian. This means that the quasi maximum likelihood estimation is efficient and consistent. Because the Gaussian state variables allow for negative interest rates and default intensities and recovery rates above 1 or below 0, it is not an ideal model; however, since the focus in this paper is on the ability to distinguish movements in the π_t-process from movements in the λ_t-process, these objections are not critical.

In the base case, all three factors are assumed to carry risk premia (as a controlling experiment the case without risk premia attached to the default and recovery risk is also considered). In theoretical terms, Bakshi et al. (2004) show that, provided investors are risk averse and recovery is stochastic, both the default intensity and the recovery rate will carry a risk premium under the pricing measure. Empirically, Driessen (2005) finds that if common factors are left out of consideration, the median company carries a premium on the credit spread risk that becomes insignificant once common factors with risk premia are added to the estimation.

Based on the result in Driessen (2005), the interest rate can be considered the model-equivalent of his common factors and should therefore carry risk premia, while the default and recovery risk factors can be considered idiosyncratic without risk premia. For this to be compatible with the insight from Bakshi, Madan, and Zhang (2004), the interest rate appears in both the default intensity and the recovery rate. This model will be denoted Model A.

By having only the interest rate as a common factor it could be argued that some systematic risk has been left unmodeled. Model B adjusts for this, and risk premia are attached to the default and recovery risk factors in addition to those already attached to the interest rate.

The structure of the assumed risk premia for Model A is given by the following change of measure

$$d\widetilde{W}_t^r = dW_t^r + (\gamma_0^r + \gamma_1^r r_t)dt.$$

The change of measure for Model B is given by

$$\begin{aligned}
d\widetilde{W}_t^r &= dW_t^r + (\gamma_0^r + \gamma_1^r r_t)dt, \\
d\widetilde{W}_t^\lambda &= dW_t^r + (\gamma_0^\lambda + \gamma_1^\lambda X_t^\lambda)dt, \\
d\widetilde{W}_t^\pi &= dW_t^r + (\gamma_0^\pi + \gamma_1^\pi X_t^\pi)dt.
\end{aligned}$$

This formulation allows for time-varying risk premia in the factor processes.

3.3 Choice of parameters

In order to simulate the model, specific parameter values are needed. This subsection will discuss the choice of parameters.

The interest rate is modeled by a single factor. Given this, the parameters chosen below are only meant to be realistic relative to the general level of interest rates. There is no intention to match the slope or other features of the yield curve documented in the vast empirical literature on yield term structures. The chosen parameters are the following

$$dr_t = 0.5(0.0375 - r_t)dt + 0.01dW_t^r.$$

Under the empirical measure, the unconditional mean is 3.75% with a volatility of 1% and a mean reversion rate of 0.5. The risk premia are assumed to be $\gamma_0^r = -1$ and $\gamma_1^r = -1$, which implies that under the Q-measure it holds that

$$dr_t = 0.49(0.0587 - r_t)dt + 0.01d\widetilde{W}_t^r.$$

The unconditional mean under the pricing measure is close to 6% with a slightly smaller mean reversion rate.

As for the factor loadings in the default intensity λ_t and the recovery rate π_t the following parameters were chosen

$$\begin{aligned} \lambda_t &= 0.01 - 0.05(r_t - 0.0375) + X_t^\lambda - 0.005, \\ \pi_t &= 0.44 + r_t - 0.0375 + X_t^\pi. \end{aligned}$$

Under the empirical measure the unconditional mean of λ_t is 1% which is close to the average one-year default probability over the business cycle of a Ba-rated company (for an example see Christensen et al. (2004)). For firms of very high credit quality (A-rated companies and above) the default intensity is so low that it is close to impossible to measure the risk contribution from the stochastic recovery.[12] On the other hand, for speculative grade companies investors are much more focused on what will be left in the event of a default, so recovery risk is believed to play a much larger role in the pricing of speculative grade bonds. Therefore, the default intensity level is matched to that of a Ba-rated company.

Empirical studies on the relationship between corporate bond spreads and treasury yields like Duffee (1998), Driessen (2005), and Bakshi et al. (2004) have all found a negative correlation between the two. One possible explanation for the negative relationship between interest rate and default intensity might be the following. The short end of the yield curve is low when the general economic climate is weak. In this case, the earnings growth in the companies will be low or negative while their coupon payments remain unchanged and the value of their debt is high. In a simple Merton-type model this will increase the probability of default, the reverse is true when the short-term interest rate level is high due to high economic activity and high earnings growth. Here, this finding is interpreted as referring mainly to the development of the default intensity, which is the reason why the factor loading on the interest rate in the default intensity function is assumed to be negative.

The chosen factor loadings in the formulation of the recovery rate π_t implies an unconditional mean recovery rate under the P-measure of 44% which is identical to the average recovery rate on senior unsecured bonds found in Moody's data as referred to by Duffee

[12]The effect of the stochastic recovery at default is too small relative to the other types of risks such as interest rate and liquidity risk to be measured accurately.

(1999) and used by Driessen (2005). Based on a sample of more than 1200 defaulted bonds and their observed recovery within one month of the default, Covitz and Han (2004) found a statistically significant positive relationship between the observed recovery rates and the yield on 3-month treasury bills.[13,14] Provided that the yield on 3-month treasury bills is a good proxy for the instantaneous interest rate r_t, it is fair to assume a positve factor loading of the interest rate in the π_t-function. As for the order of magnitude, Covitz and Han (2004) found that an increase of 1% in the 3-month treasury yield will increase the recovery rate by 3%-4%. In the simulations here, a more conservative one-to-one positive relationship is assumed.

Given that there have been no empirical studies on joint estimations of default and recovery risk, it is difficult to get a clear idea about what appropriate parameter values for the two risk factors X_t^λ and X_t^π would be. The following parameters are considered sensible choices

$$\begin{aligned} dX_t^\lambda &= 0.25(0.005 - X_t^\lambda)dt + 0.005dW_t^\lambda, \\ dX_t^\pi &= -0.25X_t^\pi dt + 0.1dW_t^\pi, \end{aligned}$$

The mean of 50 bps and the mean reversion rate of 0.25 for the X_t^λ-process is close to the median of those two parameters found by Duffee (1999)[15] when estimating the idiosyncratic factors of bond spreads across 161 companies. As for the volatility, the results are not immediately comparable since Duffee (1999) estimated a model based on CIR processes. However, taking his median estimate for the mean of 55.9 bps and for the volatility at 0.074, one can calculate a pseudo-average volatility as $0.074\sqrt{0.00559} = 0.00553$, which is close to the 50 bps used above.

For the recovery rate risk factor a similar number for the mean reversion rate has been assumed. The volatility of 10% is set in order to match the large variation in observed recovery rates found in studies like Covitz and Han (2004) and Altman and Kishore (1996). Finally, the mean has been set to 0.

Turning to the risk premia of the two factors, the following is chosen for the X_t^λ-process

$$\gamma_0^\lambda = -0.1, \quad \gamma_1^\lambda = -1.$$

It is difficult to compare these numbers to the results obtained by Duffee (1999) or Driessen

[13] They investigated both linear and non-linear relationships between the two, and in both cases they found a significant positive relationships as long as the 3-month T-Bill rate is not too high (by their numbers, below 7-8%).

[14] Bakshi et al. (2004) finds a similar positive correlation between the interest rate and market-implied recovery rates. However, in their model the effect is indirect through the impact of the interest rate on the default intensity, which in turn impacts the recovery rate. The sign, however, is not in doubt, it is positive for all 25 firms in their sample.

[15] See Table 3 in Duffee (1999) where the median κ, θ, and σ are, respectively, 0.238, 0.00559, and 0.074.

With risk premia		$P^Q(\tau > T)$	$P_{\tau>T}$	$C_{\tau>T}$	$R_{0<\tau\leq T}$	Price	Yield	Spread
$T = 1$	$C = 4\%$	0.990033	0.949385	0.038495	0.004153	0.992032	0.047684	0.005782
	$C = 7\%$			0.067366	0.004183	1.020934	0.047731	0.005842
$T = 5$	$C = 4\%$	0.950290	0.737533	0.171106	0.016259	0.924898	0.056685	0.006300
	$C = 7\%$			0.299436	0.016379	1.053348	0.056756	0.006555
$T = 10$	$C = 4\%$	0.901189	0.523835	0.293995	0.025285	0.843114	0.060290	0.006615
	$C = 7\%$			0.514491	0.025470	1.063796	0.060447	0.007064

Table 1: Basic statistics of a defaultable bond with T-year maturity and coupons paid semi-annually at rate $C = 4\%$ and $C = 7\%$, respectively. $P^Q(\tau > T)$ is the T-year survival probability under the Q-measure. $P_{\tau>T}$ and $C_{\tau>T}$ are the value of principal and coupons, respectively, conditional on survival beyond maturity. R is the recovery part of the bond price conditional on default before maturity. The parameters of the state variables are $(\kappa_r, \theta_r, \sigma_r) = (0.5, 0.0375, 0.01)$, $(\kappa_\lambda, \theta_\lambda, \sigma_\lambda) = (0.25, 0.005, 0.005)$, and $(\kappa_\pi, \theta_\pi, \sigma_\pi) = (0.25, 0, 0.1)$. The factor loadings are : $\lambda_0 = 0.01$, $\lambda_r = -0.05$, $\lambda_1 = 1$, $\pi_0 = 0.44$, $\pi_r = 1$, and $\pi_1 = 1$. Finally, the risk premia are $\gamma_0^r = -1$, $\gamma_1^r = -1$, $\gamma_0^\lambda = -0.1$, $\gamma_1^\lambda = -1$, $\gamma_0^\pi = 0.5$, and $\gamma_1^\pi = -0.5$. The values of the state variables are $r_0 = 0.0375$, $X_0^\lambda = 0.005$, and $X_0^\pi = 0$.

(2005) in their studies on corporate bond spreads, as they both work with a CIR-based model with only one idiosyncratic risk premium parameter.

The limited empirical literature does not provide any guidance on the risk premia attached to the recovery risk factor, the values below are regarded as reasonable

$$\gamma_0^\lambda = 0.5, \quad \gamma_0^\lambda = -0.5.$$

What is the impact of the risk premia? Under the equivalent pricing measure it holds that

$$\begin{aligned} dX_t^\lambda &= 0.245(0.0071 - X_t^\lambda)dt + 0.005d\widetilde{W}_t^\lambda, \\ dX_t^\pi &= 0.2(-0.25 - X_t^\pi)dt + 0.1d\widetilde{W}_t^\pi. \end{aligned}$$

The unconditional mean of the default intensity λ_t under the Q-measure will thus be 110.8 bps, while the unconditional mean of the recovery rate π_t under the pricing measure will be 21.12%.

Finally, for the corporate bonds, a set of maturities and coupon sizes have to be chosen. In order to have the short, medium, and long end of the maturity range represented, bonds with a remaining maturity of 1, 5, and 10 years will be simulated.

In their analysis of the RMV assumption, Duffie and Singleton (1999) show that default and recovery risk cannot be distinguished, only the total credit spread risk is observed. They also use numerical examples to show that for bonds trading at or very near par the identification problem is the same under the RFV assumption. For this reason the coupon sizes should be chosen so that the bonds will be trading either above or below par. Therefore, bonds with a coupon size of 4% and 7%, both paid semi-annually, are used in the simulations.

Without risk premia		$P^Q(\tau > T)$	$P_{\tau>T}$	$C_{\tau>T}$	$R_{0<\tau\leq T}$	Price	Yield	Spread
$T = 1$	$C = 4\%$	0.990273	0.949615	0.038501	0.004269	0.992385	0.047325	0.005423
	$C = 7\%$			0.067376	0.004307	1.021298	0.047368	0.005479
$T = 5$	$C = 4\%$	0.954608	0.740884	0.171407	0.018482	0.930773	0.055297	0.004911
	$C = 7\%$			0.299963	0.018645	1.059492	0.055412	0.005210
$T = 10$	$C = 4\%$	0.913360	0.530909	0.295406	0.031411	0.857725	0.058187	0.004512
	$C = 7\%$			0.516960	0.031689	1.079557	0.058474	0.005092

Table 2: Basic statistics of a defaultable bond with T-year maturity and coupons paid semi-annually at rate $C = 4\%$ and $C = 7\%$, respectively. $P^Q(\tau > T)$ is the T-year survival probability under the Q-measure. $P_{\tau>T}$ and $C_{\tau>T}$ are the value of principal and coupons, respectively, given survival beyond maturity. R is the recovery part of the bond price conditional on default before maturity. The parameters of the state variables are $(\kappa_r, \theta_r, \sigma_r) = (0.5, 0.0375, 0.01)$, $(\kappa_\lambda, \theta_\lambda, \sigma_\lambda) = (0.25, 0.005, 0.005)$, and $(\kappa_\pi, \theta_\pi, \sigma_\pi) = (0.25, 0, 0.1)$. The factor loadings are : $\lambda_0 = 0.01$, $\lambda_r = -0.05$, $\lambda_1 = 1$, $\pi_0 = 0.44$, $\pi_r = 1$, and $\pi_1 = 1$. Finally, the risk premia are $\gamma_0^r = -1$, $\gamma_1^r = -1$. The values of the state variables are $r_0 = 0.0375$, $X_0^\lambda = 0.005$, and $X_0^\pi = 0$.

Table 1 shows the prices, yields, and credit spreads[16] of the 6 different bonds tracked in the simulation when the state variables are at their unconditional mean under the P-measure. It is seen that the 7% bonds trade somewhat above par, while the 4% bonds trade well below par. Given that there are no tax or liquidity effects in the model the spreads will inevitably be below actually observed spreads of Ba-rated companies. The column denoted $R_{0<\tau\leq T}$ contains the part of the bond price that is related to the recovery at default. Its share of the total bond price ranges from 0.4% to 3%, increasing with the time to maturity. It is from this relatively modest term that the dynamics of the π_t-process are derived.

In the controlling experiment, the risk premia on the default and recovery risk are left out to examine whether this has an impact on the ability of making a joint estimation of the default and recovery risk. The statistics for the bonds at the unconditional mean of the state variables in this scenario can be found in Table 2.

Finally, to get an impression of the impact of the size of the measurement error in the observed bond yields, a normally distributed measurement error is added to the simulated bond yields. This is the model-equivalent of the problems related to bid-ask spreads and general noise in price observations. In the simulations, the benchmark will be a standard deviation of 1 bp, and a control experiment will use a standard deviation of 5 bps, which is more realistic when talking about corporate bond yields.

3.4 The simulation

The simulation of the three-factor model is performed in two steps. First, 10 years of the instantaneous interest rate process under the P-measure are simulated with monthly obser-

[16] The credit spread is defined as the difference between the yield-to-maturity of the corporate bond and that of a treasury bond with the same remaining time to maturity, coupon sizes, and payment dates.

vations starting at the unconditional mean[17]

$$r_t = \kappa_r(\theta_r - r_{t-1})\Delta t + \sigma_r\sqrt{\Delta t}z_t^r, \quad z_t^r \sim N(0,1), \quad r_0 = \theta_r,$$

where $\Delta t = \frac{1}{12}$ denotes the discrete, fixed time interval between each simulated outcome of the process. Based on the simulated path of r_t, prices of the treasury bonds with a coupon rate of 5% paid semi-annually, and maturities $\{1,2,3,5,7,10\}$[18] are calculated using the formula below

$$V^C(t,T_j) = E_t^Q[e^{-\int_t^{T_j} r_u du}] + \sum_{t_i>t}^{N} \frac{C}{2} E_t^Q[e^{-\int_t^{t_i} r_u du}]$$

and converted into yields to maturity by

$$\sum_{t_i>t}^{N} \frac{C}{2} e^{-y^C(t,T_j)t_i} + e^{-y^C(t,T_j)T_j} = V^C(t,T_j).$$

A normally distributed measurement error with mean zero and a standard deviation of 1 bp is added to these treasury yields. This makes up the 'observed' set of treasury bond yields, which will be common to all firms in the economy.

Next, for each i of a total of 50 firms, 10 years of the $X_t^{i,\lambda}$- and $X_t^{i,\pi}$-processes are simulated in a similar fashion to r_t under the P-measure

$$\begin{aligned}
X_t^{i,\lambda} &= \kappa_\lambda^i(\theta_\lambda^i - X_{t-1}^{i,\lambda})\Delta t + \sigma_\lambda^i\sqrt{\Delta t}z_t^{i,\lambda}, \quad z_t^{i,\lambda} \sim N(0,1), \quad X_0^{i,\lambda} = \theta_\lambda^i, \\
X_t^{i,\pi} &= \kappa_\pi^i(\theta_\pi^i - X_{t-1}^{i,\pi})\Delta t + \sigma_\pi^i\sqrt{\Delta t}z_t^{i,\pi}, \quad z_t^{i,\pi} \sim N(0,1), \quad X_0^{i,\pi} = \theta_\pi^i.
\end{aligned}$$

Prices of defaultable bonds, with two different coupon rates, 4% and 7%, paid semi-annually and maturities $\{1,5,10\}$, are calculated, converted into yields, and a measurement error with a mean of zero and a standard deviation of 1 bp (and 5 bps in the control experiment) is added.

In total, 4 sets of observed corporate bond yields are calculated for each set of simulated paths, $(r_t, X_t^{i,\lambda}, X_t^{i,\pi})$, and they are as follows

(i). Without risk premia attached to $(X_t^{i,\lambda}, X_t^{i,\pi})$ and a standard deviation of the measurement error of 1 bp (Model A, $\sigma_\varepsilon = 1$ bp).

(ii). With risk premia attached to $(X_t^{i,\lambda}, X_t^{i,\pi})$ and a standard deviation of the measurement error of 1 bp (Model B, $\sigma_\varepsilon = 1$ bp).

(iii). Without risk premia attached to $(X_t^{i,\lambda}, X_t^{i,\pi})$ and a standard deviation of the measurement error of 5 bps (Model A, $\sigma_\varepsilon = 5$ bps).

[17] The diffusion processes are simulated using the Euler method.

[18] The set of maturities is kept fixed out of computational convenience. However, introducing maturity reduction and on-going new issuances of matured bonds is not believed to influence the results in any significant way.

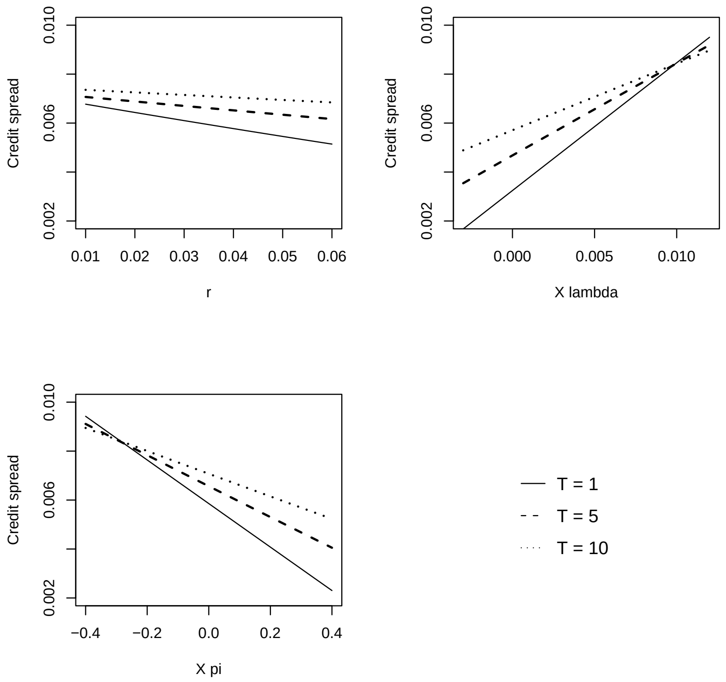


Figure 1: Illustration of the sensitivity of the credit spread of the defaultable 7% coupon bonds to changes in the three risk factors, r_t, X_t^λ, and X_t^π, respectively. The full-drawn lines are for the 1-year maturity. The dashed and dotted lines are for the 5- and 10-year maturities, respectively. In the base case the parameter values are $(\kappa_r, \theta_r, \sigma_r) = (0.5, 0.0375, 0.01)$, $(\kappa_\lambda, \theta_\lambda, \sigma_\lambda) = (0.25, 0.005, 0.005)$, and $(\kappa_\pi, \theta_\pi, \sigma_\pi) = (0.25, 0, 0.1)$. The factor loadings are $\lambda_0 = 0.01$, $\lambda_r = -0.05$, $\lambda_1 = 1$, $\pi_0 = 0.44$, $\pi_r = 1$, and $\pi_1 = 1$. Finally, the risk premia are $\gamma_0^r = -1$, $\gamma_1^r = -1$, $\gamma_0^\lambda = -0.1$, $\gamma_1^\lambda = -1$, $\gamma_0^\pi = 0.5$, and $\gamma_1^\pi = -0.5$. The values of the state variables are equal to their unconditional mean under the P-measure $r_0 = 0.0375$, $X_0^\lambda = 0.005$, and $X_0^\pi = 0$.

(iv). With risk premia attached to $(X_t^{i,\lambda}, X_t^{i,\pi})$ and a standard deviation of the measurement error of 5 bps (Model B, $\sigma_\varepsilon = 5$ bps).

The first two cases that have a measurement error with a standard deviation of 1 bp ($\sigma_\varepsilon = 1$ bp) will be considered the benchmark cases, in the sense that they reveal what can be obtained under close to ideal conditions.

3.5 The estimation

Once the simulated data for each firm is obtained, a quasi maximum likelihood method based on the Kalman filter is used to estimate the paths and parameters for the three risk factors. The estimation is performed in two steps. First, the parameters and the path of the interest rate process are estimated from the simulated treasury bond yields using a standard extended Kalman filter. Since the simulated path of the interest rate is common to all 50 firms, this step only has to be performed once. Second, for each firm, the parameters and paths of the idiosyncratic risk factors, $X_t^{i,\lambda}$ and $X_t^{i,\pi}$, including all the factor loadings, are estimated using

ψ	true value	estimate
κ_r	0.5	0.500
θ_r	0.0375	0.0387
σ_r	0.01	0.00942
γ_0^r	-1	-0.998
γ_1^r	-1	-1.01
σ_ε^r	0.0001	0.0000980

Table 3: Result of the estimation of the interest rate process r_t based on the simulated yields for the treasury bonds.

the quasi maximum likelihood of the extended Kalman filter. Appendix C has the technical details including a discussion of the identification problem involved in the second step.

3.5.1 Factor sensitivity of the credit spread

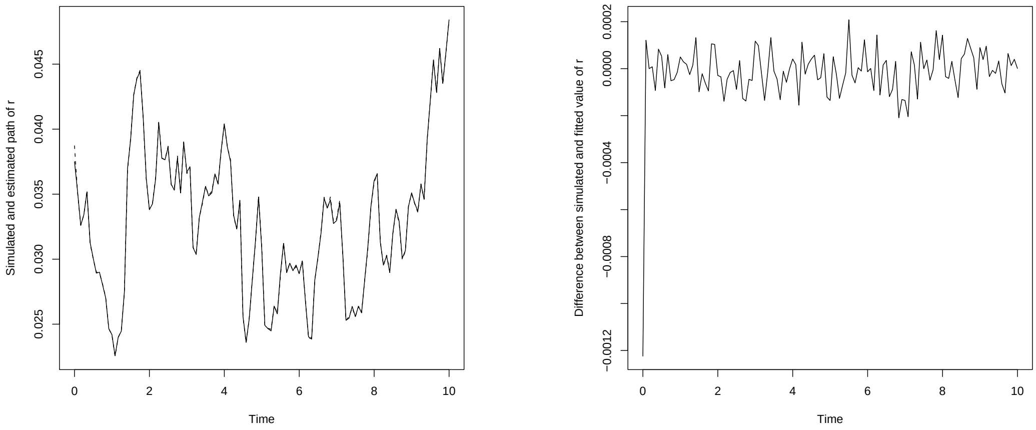
This subsection examines the sensitivity of the credit spread of defaultable bonds to changes in the risk factors and will shed some light on the possibility of estimating a model of defaultable debt that incorporates both recovery risk and default intensity risk.

To illustrate the effect of varying the values of the state variables, start out by fixing a set of parameter values and factor loadings and let the state variables assume their unconditional mean under the P-measure. This produces the base case credit spreads for the 1-, 5-, and 10-year maturity bonds in Table 1. Next, each state variable is varied in an interval around its unconditional mean while keeping the other two state variables unchanged and the corresponding credit spread for the 3 different maturities is calculated. For the 7% coupon bonds, this yields the three graphs in Figure 1.

Across maturities, the smallest effects are observed for the bonds with longer maturities which is caused by the mean-reversion of the state variables. The implication of this is that for very long maturities the current values of the state variables have only marginal impact on the corporate bond yield spread. On the other hand, for bonds with shorter maturities the spread is much more sensitive to changes in the state variables. In order to pin down the path of the risk factors, X_t^λ and X_t^π, it is necesasry to have both short and long maturity bonds represented in addition to different coupon sizes for each maturity.

It is worth noticing that the credit spread varies approximately linearly in all three factors for each of the three maturities. This means that if one uses the extended Kalman filter method for estimating a model with default intensity and recovery risk, the common practice of linearizing the measurement equation through a first-order Taylor expansion is a satisfactory approximation for estimation purposes.[19]

[19]For further details see Appendix B and for applications of this method to real corporate bond data see Duffee (1999) and Driessen (2005).



(a) The simulated (solid line) and estimated (dashed line) path of the interest rate process r_t.

(b) The difference between the simulated path of r_t and the estimated r_t-values from the extended Kalman filter estimation.

Figure 2: Result of extended Kalman filter estimation based on the simulated treasury yields.

4 Results

This section goes through the results of the estimations based on the simulated data described in Section 3. The result of the Kalman filter estimation based on the simulated treasury bond yields is described in Section 4.1. Section 4.2 contains a thorough analysis of the results of the subsequent joint estimation of default and recovery risk and in Section 4.3 the identification problem that causes problems in the joint estimation is discussed.

4.1 The Kalman filter estimation of the interest rate r_t

In Table 3 the true and estimated parameter values of the interest rate process are listed. Because the requirement of normally distributed errors in the affine state equation and the measurement equation are both satisfied by the simulated treasury yields, it is well known that the Kalman filter is an efficient and consistent estimator of the true parameters (see Hamilton (1994), Section 13.4). The result in Table 3 is a reflection of this fact with only minor differences between the estimated and the true values.

The simulated and estimated path of r_t are illustrated in Figure 2(a) and are practically identical, Figure 2(b) depicts the difference between the two paths. The mean absolute error of the estimated path is equal to 6.61×10^{-5} or little more than half of the standard deviation of the included measurement error.[20] A standardized measure of the difference between the two paths is given by the mean absolute error relative to the conditional standard deviation

[20]Note that a perfect fit is never obtained as yields are observed with error.

ψ	true value	mean (A)	median(A)	sd(A)	mean (B)	median(B)	sd(B)
λ_0	0.01	0.00994	0.00996	0.000440	0.00996	0.00995	0.000812
λ_r	-0.05	-0.0499	-0.0500	0.00390	-0.0498	-0.0501	0.00385
λ_1	1	1.11	1.07	0.234	1.02	1.02	0.138
π_0	0.44	0.438	0.438	0.0263	0.446	0.442	0.0376
π_r	1	1.03	1.05	0.415	1.02	1.02	0.473
π_1	1	0.805	0.946	0.429	0.951	0.970	0.142
κ_λ	0.25	0.250	0.253	0.0109	0.247	0.246	0.0108
σ_λ	0.005	0.00481	0.00492	0.000813	0.00505	0.00504	0.000841
κ_π	0.25	0.248	0.247	0.0210	0.270	0.257	0.0474
σ_π	0.1	0.0905	0.0897	0.0309	0.100	0.100	0.00139
γ_0^λ	-0.1				-0.104	-0.0999	0.0310
γ_1^λ	-1				-0.799	-0.875	0.568
γ_0^π	0.5				0.522	0.502	0.0681
γ_1^π	-0.5				-0.681	-0.521	0.493
σ_ε	0.0001	0.0000994	0.0000993	4.16×10^{-6}	0.000100	0.0000996	2.89×10^{-6}

Table 4: Results of the estimations based on simulated data with a measurement error with mean zero and a standard deviation of $\sigma_\varepsilon = 1$ bp for all 50 firms. 'A' refers to the simulations where the idiosyncratic risk factors do not carry any risk premia, while 'B' refers to the simulations where they do carry risk premia.

of the interest rate process over time intervals of length Δt, i.e.

$$\frac{1}{\frac{1}{\Delta t}N}\sum_t \frac{|r_t - \widehat{r}_t|}{\sqrt{V_t[r_{t+\Delta t}]}}.$$

With a value of 0.0233 for the estimated $\widehat{r}_t$-path this measure indicates that the average error is less than 2.5% of the random noise induced by the Brownian motion. This means that errors caused by the Kalman filter are insignificant for practical purposes. The above result will serve as a benchmark of what can be achieved under ideal conditions.

4.2 The Kalman filter estimation of default and recovery risk

Table 4 presents the empirical mean, median, and standard deviation of the parameter estimates for the 50 simulated firms[21] for the two benchmark cases (Section 3.4, (i-ii)) where the measurement error has a standard deviation of 1 bp. Model A refers to the simulated data without any risk premia attached to the default and recovery risk factors, $X_t^{i,\lambda}$ and $X_t^{i,\pi}$, while Model B refers to the simulated data where such risk premia have been attached to these risk factors.

In comparing the two benchmark cases, Model A and Model B, the ability to separate the default intensity process from the recovery rate process is examined; the uncertainty of the

[21] Due to the time required for each optimization only 50 simulations have been performed for each type of experiment. Performing further simulations is not believed to add much insight into the problem of making a joint estimation of default and recovery risk.

estimates, proximity of their mean and median to the true values, proximity of the estimated path to the simulated path, and precision of the risk premia are all considered. Based on the preceding criteria it is concluded that overall Model B (which includes risk premia) performs better.

An initial intuitive question when decomposing bond spreads into a default component and a recovery component is whether the model is able to accurately estimate the mean of each component. As for the dynamics under the P-measure, this question can be answered simply by looking at the estimates of the constant terms in the default intensity, λ_0, and in the recovery rate, π_0. In both models these terms have mean and median close to the true values, and the histograms of the estimate distributions indicate no bias.[22] The uncertainty about the estimates, as measured by their standard deviation, increases when moving from Model A to Model B, but does not reach unreasonable levels. Pan and Singleton (2005) make a simulation study based on CDS spreads somewhat along the same lines as the current study. For the default intensity the two studies are not comparable, as Pan and Singleton (2005) use square-root (CIR), lognormal, and three-halves processes, however concerning the recovery rate they use a constant rate fixed at 0.25.[23] Thus, the results for the π_0-parameter in Table 4 can, with some caution, be compared to the accuracy with which Pan and Singleton (2005) estimate their constant recovery. When they simulate a stationary λ_t-process under the Q-measure (which is the appropriate comparison to the current study), their mean estimate of the recovery rate is upward biased (0.285 vs. the true 0.25), but the empirical standard deviation of their 100 estimates is 0.0135. With a standard deviation of the π_0-estimates of 0.0263 and 0.0376 in Model A and B, respectively, the accuracy is somewhat less, but the estimates appear to be unbiased.[24]

The case of allowing for factors derived from alternative data sources, represented in the current framework by the interest rate process r_t estimated from treasury yields, is relevant for practical applications of the model developed in this paper. Examining the ability of the models to distinguish the factor loadings in the default intensity and the recovery rate of such exogenously given factors is critical. The numbers for the factor loadings λ_r and π_r in Table 4 show that Model A and Model B perform equally well in this respect.

The mean and median of λ_r in both models are very close to the true value and their standard deviations are almost identical. The histograms for the 50 estimates of λ_r in Models A and B[25] are consistent with this observation in that the bulk of the estimates are very close to the true factor loading.

For the factor loading in the recovery rate π_r, despite the fact that Model A has a slightly

[22]See Figures 6 and 9.

[23]The recovery rate of 25% appears as a loss rate of 75% in their model.

[24]Unfortunately, it is not possible to tell what the level of the standard deviation of the measurement error is in their study, as their value for σ_ε of 0.5 refers to the standard deviation relative to the absolute size of the bid-ask spread in their real CDS data.

[25]See Figures 6 and 9.

smaller standard deviation, its mean and median are farther from the true value than that of Model B. Looking at the empirical distribution of the π_r-estimates[26] it is observed that the π_r-estimate of Model A is upward biased. In addition, both models have fat tails indicating that estimates anywhere in the interval between 0 and 2 cannot be excluded. Therefore, if Model A were to be chosen, even though there would be a slightly smaller uncertainty about the estimate, on average it would be further from the true value.

A true separation of default intensity and recovery rate can only be said to be successful, if the model is accurate in the decomposition of bond yield movements into movements of the default intensity and movements of the recovery rate. The current model's level of success can be measured by examining the following:

- The estimates of the factor loadings, λ_1 and π_1, of the default and recovery risk factors.
- The ability of the model to estimate the dynamic properties of $X_t^{i,\lambda}$ and $X_t^{i,\pi}$.
- The proximity of the estimated paths of $X_t^{i,\lambda}$ and $X_t^{i,\pi}$ to the simulated ones.

Each of these three measures will be studied in turn.

Concerning the factor loadings, there is a clear difference in the performance between the two models. The standard deviation for the factor loading λ_1 of X_t^{λ} in Model B is only one-half that of Model A. The standard deviation for the factor loading π_1 of X_t^{π} in Model B is only one-third that of Model A.

For both models, the distribution of the factor loading for the default intensity risk factor λ_1 is positively skewed with a mean above the true value. The distribution of the factor loading of the recovery risk factor π_1 is negatively skewed with a mean below the true value[27] and this is more pronounced for Model A than for Model B. This implies that the estimate of λ_1 is typically above the true value, and the impact of idiosyncratic changes to λ_t are overestimated. The estimate of π_1 has a tendency to be below its true value, and the impact of idiosyncratic changes to π_t tend to be underestimated. Part of the relatively large standard deviation for these parameters may be explained by the manner in which the model has been identified by fixing the mean of the two processes, θ_λ and θ_π, respectively, and leaving λ_1 and π_1 as free parameters (see Appendix B for details). The uncertainty of the λ_1- and π_1-estimates can be seen as a reflection of the usual difficulty of estimating the mean parameter of a process with a relatively low rate of mean reversion given only 10 years of data.

With respect to the dynamic properties of the default and recovery risk processes, $X_t^{i,\lambda}$ and $X_t^{i,\pi}$, the numbers provided in Table 4 resemble to a large extent the findings in Duffee and Stanton (2004), who also perform Monte Carlo simulation studies with subsequent estimations based on the Kalman filter, amongst other methods.[28] For the one-factor Gaussian term

[26] See Figures 7 and 10.

[27] See Figures 6, 7, 9, and 10.

[28] In most cases they make 500 simulated sets of yield observations for either 2 or 5 zero-coupon bonds with 1000 weekly observations for each bond.

structure model with affine price of risk (the equivalent of the interest rate model applied in this paper), they find that the estimate of the mean-reversion rate κ under the physical measure is biased, while the estimate of κ under the Q-measure is unbiased. They note that, primarily, this phenomenon is a small-sample property that is particularly strong for near unit-root processes, and thus this finding is not limited to, and certainly not caused by, the Kalman filter. With a constant risk premium, which implicitly includes the case of no risk premia, they do not find any such bias.

With flexible dynamics of the risk premia, the risk factor has the following properties under the physical and the equivalent measure[29]

- Under the physical measure: $dX_t^\lambda = (\kappa_\lambda\theta_\lambda - \kappa_\lambda X_t^\lambda)dt + \sigma_\lambda dW_t^\lambda$.
- Under the Q-measure: $dX_t^\lambda = (\kappa_\lambda\theta_\lambda - \gamma_0^\lambda\sigma_\lambda - (\kappa_\lambda + \gamma_1^\lambda\sigma_\lambda)X_t^\lambda)dt + \sigma_\lambda d\widetilde{W}_t^\lambda$.

This shows that the physical and risk-neutral drift terms do not share any common parameters due to the flexibility of the risk premium structure. As bond yields are priced under the Q-measure, there is no bias in the estimate of the mean-reversion rate under the Q-measure, $\kappa_\lambda^Q = \kappa_\lambda + \gamma_1^\lambda\sigma_\lambda$. However, there will be a bias in the estimate of the mean-reversion rate under the P-measure. Given the above relationship between the mean-reversion rate under the two measures, those two features are not compatible unless the estimate of the risk premium parameter, γ_1^λ, is biased in the opposite direction of the estimate of κ_λ.

Consistent with the results in Duffee and Stanton (2004), the results for Model A, the model without risk premia, indicate that there is no bias in either the κ_λ- or the κ_π-estimates (see Table 4). However, a downward bias in the σ_π-estimate in Model A is observed, which is inconsistent with the findings in Duffee and Stanton (2004), where they report no bias for volatility estimates in any of their Gaussian one- or two-factor models.

For Model B, the model with the flexible risk premium structure, there is only weak evidence of a downward bias in the κ_λ-estimates, however, there is bias in the corresponding risk premium γ_1^λ-estimate as expected. Thus, the seeming lack of bias in the κ_λ-estimates can be explained by the small size of the product $\gamma_1^\lambda\sigma_\lambda$ relative to the size of κ_λ, the risk premium is simply not large enough to induce an observable bias in κ_λ. On the other hand, for the κ_π-estimates there is a clear upward bias similar to the finding of Duffee and Stanton (2004). And the corresponding risk premium, γ_1^π, is downward biased.

For the constant part of the risk premia, γ_0^λ and γ_0^π, no bias is observed, and combined with a reasonable level of the standard deviations this is in line with what would be expected based on the results in Duffee and Stanton (2004). As their κ-values are much smaller than the ones used in this paper (and therefore closer to the unit-root case), they find evidence of 'strong bias', while the combinations of $(\kappa, \sigma, \gamma_1)$-values considered in this paper only give

[29] Here, the default intensity risk factor is used just as an example, the same argument holds for the recovery risk factor X_t^π.

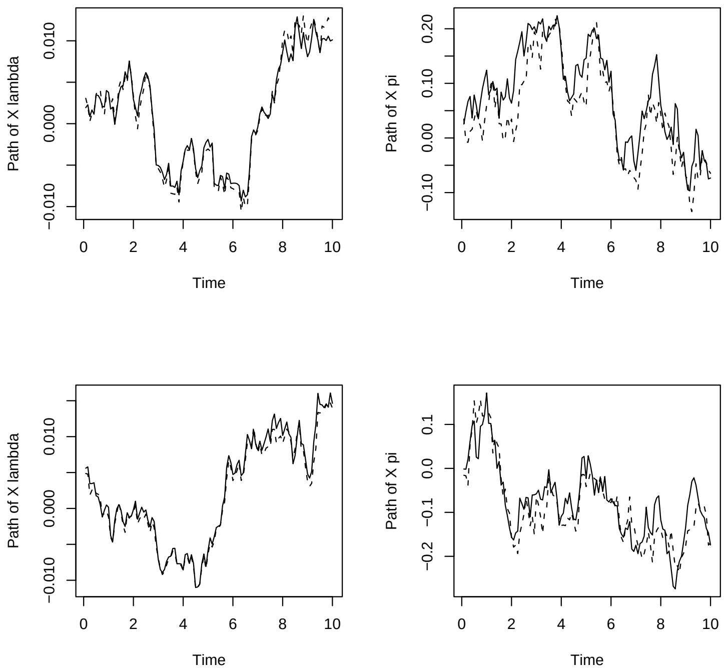


Figure 3: Illustration of the simulated paths for X_t^{λ}, and X_t^{π} along with the corresponding implied paths from the Kalman filter estimation based on simulated yields containing risk premia for both the default and the recovery risk factors (Model B). The standard deviation of the measurement error is 1 bp. The simulated paths are the solid lines, whereas the estimated paths are given by the dashed lines.

rise to relatively modest levels of bias, in particular if compared to the standard deviation of the estimates.

If out of convenience (or lack of data in an empirical setting) one or more common factors are neglected, risk premia will need to be attached to the idiosyncratic risk factors as demonstrated by Driessen (2005). The results in Table 4 show that adding risk premia introduces a bias in the estimation of the mean-reversion rates of the default and recovery risk factors and a bias in the estimation of the stochastic part of the corresponding risk premium. However, this does not take away precision from the other parameter estimates, in fact, it might be argued that it adds precision for several relevant parameters, most notably λ_1, π_1, and σ_{π}.

Thus far the question has been how well the parameters are estimated. However, another equally important dimension is how well the models are performing in terms of the accuracy with which the estimated paths mimic the simulated paths. One advantage of the Kalman filter method is that, in addition to parameter estimates, it also delivers the corresponding most likely path of the state variables. So how well are the estimations doing with respect to this measure? In Figure 3 the estimated paths of $X_t^{i,\lambda}$ and $X_t^{i,\pi}$ are illustrated for two representative sets of Model B simulated data. In general, neither of the two paths are matched as closely as in the estimation of the interest rate process in Section 4.1. For the

	mean (A)	median(A)	sd(A)	mean (B)	median(B)	sd(B)
$X_t^{i,\lambda}$	0.874	0.698	0.680	0.791	0.602	0.504
$X_t^{i,\pi}$	11.6	1.97	30.4	1.57	1.31	0.668

Table 5: Results for the standardized mean absolute error of the estimated paths of $X_t^{i,\lambda}$ and $X_t^{i,\pi}$, which is calculated as $\frac{1}{\frac{1}{\Delta t}N}\sum_t \frac{|X_t^{i,\lambda}-\widehat{X}_t^{i,\lambda}|}{\sqrt{V_t[X_{t+\Delta t}^{i,\lambda}]}}$ and $\frac{1}{\frac{1}{\Delta t}N}\sum_t \frac{|X_t^{i,\pi}-\widehat{X}_t^{i,\pi}|}{\sqrt{V_t[X_{t+\Delta t}^{i,\pi}]}}$, respectively, based on 50 estimations. 'A' refers to the simulations where the idiosyncratic risk factors do not carry any risk premia, while 'B' refers to the simulations where they do carry risk premia. The standard deviation of the measurement errors were set to 1 bp.

default intensity risk factor, the estimated paths track the simulated paths reasonably well with minor temporary deviations. For the estimated paths of the recovery risk factor the temporary deviations tend to be relatively larger and last for a longer period of time.

In order to compare the deviations across processes a standardized measure of the deviation of each estimated path is needed. The preferred statistic used in this paper is given by calculating, for each observation point, the absolute error relative to the conditional standard deviation of the process; the mean of these ratios along the entire estimated path is then calculated and regarded as the standardized measure. Table 5 shows the mean, median, and standard deviation of calculating the standardized mean absolute error for each of the estimated paths in the 50 estimations for Model A and B, respectively. Judged by this statistic, Model B with risk premia attached to the default and recovery risk factors is better at jointly estimating both the default intensity risk and the recovery risk. In terms of the level of error in the Model B-estimations, the mean estimate of the average absolute error of the estimated path of the default intensity risk factor is 0.79 times the conditional standard deviation of the $X_t^{i,\lambda}$-process. Given the parameters of the $X_t^{i,\lambda}$-process used in the simulation, the conditional one-month standard deviation is 14.3 bps, so the absolute size of the average error is approximately 10-12 bps. For the recovery risk factor in Model B the statistic is close to 1.6 which means an average absolute error in the estimated path values of the state variable of 4-5%[30] which is less satisfactory. For the Model A-estimations, the result for the estimated paths of the default risk factor, $X_t^{i,\lambda}$, is only marginally worse than those found in the Model B-estimations. However, for the estimated paths of the recovery risk factor $X_t^{i,\pi}$ the statistic is significantly worse in the Model A-estimations with a median of almost 2, and a mean of 11.6.[31]

The overall conclusion is that both models can claim some success in estimating the path of the default risk factor $X_t^{i,\lambda}$. However, for the recovery risk factor $X_t^{i,\pi}$, it is only Model B (that with risk premia attached to both risk factors) that can claim success in closely replicating the simulated paths.

[30] The 1-month conditional standard deviation of $X_t^{i,\pi}$ is 2.86% at the true parameters.

[31] This is clearly influenced by a handful of outliers where the estimated path of $X_t^{i,\pi}$ is far from the simulated path.

ψ	λ_0	λ_r	λ_1	κ_λ	σ_λ	π_0	π_r	π_1	κ_π	σ_π	σ_ε
λ_0	1	-0.174	-0.327	-0.820	0.265	0.860	0.088	0.058	0.715	0.502	-0.011
λ_r		1	0.198	-0.090	-0.383	0.065	0.008	-0.323	0.175	-0.251	-0.051
λ_1			1	0.327	-0.877	-0.272	-0.057	-0.156	-0.132	-0.678	-0.121
κ_λ				1	-0.151	-0.758	-0.137	0.097	-0.615	-0.456	-0.042
σ_λ					1	0.152	0.072	0.194	0.026	0.646	0.097
π_0						1	-0.306	0.190	0.729	0.474	-0.006
π_r							1	-0.503	0.067	-0.036	-0.020
π_1								1	-0.040	0.120	0.011
κ_π									1	0.385	-0.063
σ_π										1	-0.079
σ_ε											1

Table 6: Matrix of correlation coefficients for the empirical distribution of parameter estimates in Model A without risk premia on the default risk and the recovery risk factor. The standard deviation of the true measurement error was set equal to 1 bp. The matrix has been calcuted based on the results of 50 estimations.

The above analysis of the estimated parameters has been partial, in the sense that the properties of each parameter have been judged by examining the marginal distributions. One way to detect systematic patterns across parameter estimates is to look at the pairwise correlation coefficients. With 50 estimations in each model, there are not enough observations to obtain a stable estimate of the matrix of pairwise correlation coefficients. Efron (1982) talks about 1000 simulations or more needed to get a good picture of the tails of a distribution, and it seems that something similar holds for moments of higher order.

For completeness, the empirical matrices of correlation coefficients are presented in Table 6 for Model A and Table 9 for Model B. When analyzing the correlations, the estimate for the standard deviation of the measurement error σ_ε is consistently close to being uncorrelated with the other parameter estimates.

In summarizing the findings for the benchmark cases, those with 1 bp measurement noise standard deviation, Model B dominates. It is noted that in both models the estimation is able to accurately determine the mean of the default intensity and of the recovery rate, this holds in estimating the factor loadings for exogenously given factors as well. But when it comes to estimating the characteristics of the default and recovery risk factors, Model B performs better, despite the bias imposed on the estimation of the mean-reversion parameters by the flexibility of the affine risk premia structure. For important parameters like λ_1, π_1, and σ_π, the estimates in Model B are closer to the true value and have smaller standard deviations and the deviations of the estimated paths from the true paths are smaller than those observed in Model A. Finally, the risk premia themselves are very important (which Model B incorporates); the price of default and recovery risk is of interest in a wide range of applications.

ψ	true value	mean (A)	median(A)	sd(A)	mean (B)	median(B)	sd(B)
λ_0	0.01	0.00962	0.00958	0.00122	0.0102	0.0101	0.00198
λ_r	-0.05	-0.0387	-0.0361	0.0490	-0.0470	-0.0490	0.0370
λ_1	1	0.991	1.00	0.195	1.04	1.05	0.247
π_0	0.44	0.434	0.443	0.0509	0.425	0.428	0.0818
π_r	1	0.862	1.09	1.82	0.864	0.998	1.24
π_1	1	0.793	0.859	0.992	0.982	1.02	0.396
κ_λ	0.25	0.252	0.250	0.0313	0.245	0.243	0.0297
σ_λ	0.005	0.00525	0.00508	0.000922	0.00505	0.00487	0.00141
κ_π	0.25	0.401	0.264	0.739	0.329	0.280	0.197
σ_π	0.1	0.0979	0.0950	0.0995	0.0839	0.0922	0.0356
γ_0^λ	-0.1				-0.0997	-0.0873	0.0934
γ_1^λ	-1				-0.738	-0.903	1.55
γ_0^π	0.5				0.614	0.561	0.327
γ_1^π	-0.5				-1.13	-0.675	1.95
σ_ε	0.0005	0.000499	0.000496	1.52×10^{-5}	0.000498	0.000495	1.38×10^{-5}

Table 7: Result of the estimations based on simulated data with a measurement error with zero mean and a standard deviation of $\sigma_\varepsilon = 5$ bps for a total of 50 firms. 'A' refers to the simulations where the idiosyncratic risk factors do not carry any risk premia, while 'B' refers to the simulations where they do carry risk premia. For each parameter in each type of model the empirical mean, median, and standard deviation are provided.

4.2.1 Control experiment: standard deviation of measurement error fixed at 5 bps

This subsection will briefly go through the major findings of the control experiment, where the noise terms added to the simulated yields have a standard deviation of 5 bps, instead of the 1 bp considered in the preceding section.

Given the noise and lack of highly liquid corporate bonds for many corporate bond issuers, the results discussed so far can be viewed as a benchmark of what can be achieved under ideal conditions. By increasing the volatility of the measurement error, the experimental environment approaches that which is observed in the market. The analysis is not performed with a noise volatility of, say 25 bps or more (for the given set of parameters), as the finer details of the recovery process would surely drown in this sea of noise.[32]

Table 7 shows the result of the estimations based on data with a standard deviation of the measurement error of 5 bps. As before, Model A refers to estimations based on simulated data without risk premia attached to the default and recovery risk factors, $X_t^{i,\lambda}$ and $X_t^{i,\pi}$, while Model B refers to the result of estimations where such risk premia are included in those two processes.

The fundamental question remains the same: Is the model able to accurately decompose

[32]If the mean level of the default intensity was raised to, say 300 bps, the importance of the recovery component would increase correspondingly, thus an estimation at higher noise levels is not excluded *per se*, but in the given parameter setting to go beyond 5-10 bps noise volatility does not make sense, as demonstrated by the results in this section.

	mean (A)	median(A)	sd(A)	mean (B)	median(B)	sd(B)
$X_t^{i,\lambda}$	1.31	1.15	0.687	1.72	1.43	1.01
$X_t^{i,\pi}$	20.5	2.86	94.2	3.53	2.85	1.65

Table 8: Results for the standardized mean absolute error of the estimated paths of $X_t^{i,\lambda}$ and $X_t^{i,\pi}$, which is calculated as $\frac{1}{\frac{1}{\Delta t}N}\sum_t \frac{|X_t^{i,\lambda}-\widehat{X}_t^{i,\lambda}|}{\sqrt{V_t[X_{t+\Delta t}^{i,\lambda}]}}$ and $\frac{1}{\frac{1}{\Delta t}N}\sum_t \frac{|X_t^{i,\pi}-\widehat{X}_t^{i,\pi}|}{\sqrt{V_t[X_{t+\Delta t}^{i,\pi}]}}$, respectively. 'A' refers to the simulations where the idiosyncratic risk factors do not carry any risk premia, while 'B' refers to the simulations where they do carry risk premia. The standard deviation of the measurement errors were set to 5 bps.

the observed bond yields into a default component and a recovery component?

With respect to the mean under the P-measure, represented by the λ_0- and the π_0-parameters, both models still accurately estimate the mean of the default intensity. As far as the mean of the recovery rate is concerned, Model B performs worse given the increased noise, with a downward biased mean estimate and a volatility that has more than doubled.

The ability to accurately separate the impact of exogenous factors like the interest rate r_t is reduced. For λ_r and π_r, in both models, there is now a clear downward bias in the mean estimate, and in three out of the four cases, the standard deviation is now larger than the mean of the estimated parameters.

In terms of the ability to estimate the factor loadings and dynamic properties of the default and recovery risk factors, there is a divide in the results (shown in Table 7). For the default risk factors in Model A, the standard deviation of the estimate for λ_1 actually decreases and the uncertainty about the σ_λ-estimate only grows marginally, while the standard deviation of the κ_λ-estimate is tripled, the estimate is still unbiased. In Model B, the downward bias in κ_λ and the upward bias in κ_π becomes more pronounced while the standard deviation of those same parameter estimates increases by a factor 3 and 4, respectively. A natural consequence of this is that the biasness and the standard deviations of the risk premium parameters, γ_1^λ and γ_1^π, have changed in a similar way. Overall, for the recovery risk factor in both models, the performance of the estimation has deteriorated. This finding shows that, given increased observation noise, it is still reasonable to estimate the properties of the default intensity, but for the recovery rate the properties of the process are in general not determined with as much accuracy as would be observed in a setting with less noise.

As for the estimated paths of the risk factors, much the same results are found. Examining the standardized measure of the deviations of the estimated paths from the simulated ones, shown in Table 8, it is observed that the size of the measure has doubled, but Model A performs better with respect to tracking the path of the default intensity risk factor $X_t^{i,\lambda}$, while Model B remains the better for tracking the path of the recovery risk factor $X_t^{i,\pi}$.

The general conclusion derived from this experiment is that, naturally, as observation noise increases the uncertainty about the parameter estimates grows, but the model's ability to estimate the essential parameters of the default and recovery risk factors seems to decline somewhat beyond what one would have expected from a moderate sized noise volatility of 5 bps.[33]

The next section will discuss the nature and cause of the model's weaker performance as revealed by the experiment with the 5 bps noise volatility.

4.3 Identification problem

The difficulties in the estimations above can be traced to the problem of precisely determining the estimated paths of the default and recovery risk factors. This is caused by a fundamental identification problem inherent in the corporate bond yields as soon as measurement noise is added to the true yields. The basic feature of this phenomenon is illustrated in Figure 4.

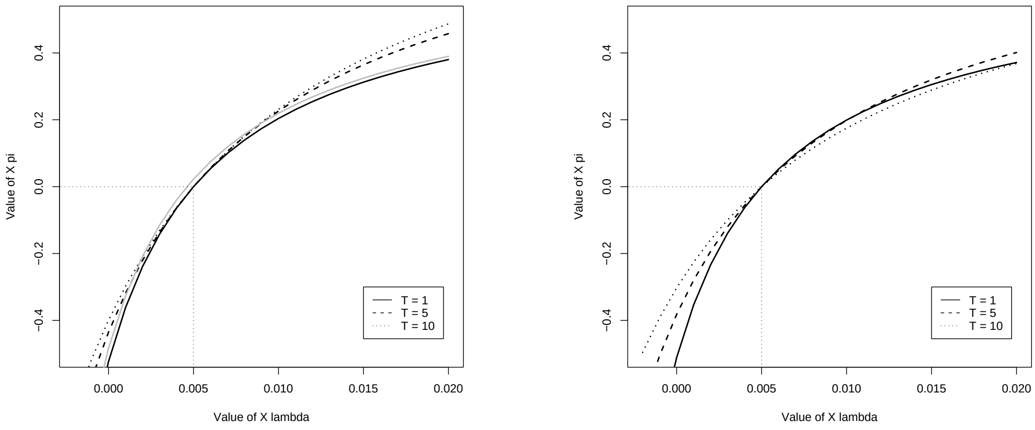
Figures 4(a) and 4(b) show the combinations of (X_t^λ, X_t^π) that would give a perfect fit of the true yields (evaluated at the unconditional mean of each process) for the three observed maturities for the 7% and the 4% coupon bonds, respectively.[34] From these figures it follows that without any measurement error, there is a unique choice that gives a perfect fit, namely the original simulated value. However, as soon as measurement noise is added to the picture, exemplified in Figure 4(a) by a negative error of -2 bps to the observed yield of the 1-year 7% bond, there is no longer a unique combination that gives a perfect fit. Instead, the Kalman filter tries to find the choice of (X_t^λ, X_t^π) that gives the smallest sum of squared errors across the 6 observed bond yields while taking the underlying dynamics of the two processes into account. From the figure it is seen that this balancing procedure (minimizing squared errors vs. being loyal to the dynamics of the state variables) will result in a tendency to either exceed or understate the estimated path values of both risk factors.[35] This pattern can be observed in Figure 3 which illustrates examples of the estimated paths against the simulated ones.

This feature of corporate bond yields makes one look for other financial products that allow the joint estimation of default and recovery risk with greater precision than found in the study of simulated corporate bond yields. One important and relevant alternative would be the credit default swap. The market for this product has been growing very rapidly in recent years, by some measures it is more liquid than the corporate bond market, and there are indications that the default swap market leads the corporate bond market in terms of incorporating the most recent information (see Blanco, Brennan, and Marsh (2004) for details). Could these products be used to help solve the identification problem found when

[33] To put the 5 bps into perspective it can be noted that Kim (2004) performs simulations with 15 bps noise volatility, while Duffee and Stanton base their simulations on 60 bps error volatility.

[34] Kim (2004) has an extensive discussion of the identification problem in the quadratic yield term structure models. There, the state variables that give a perfect fit to observed yields take the form of an ellipse. For the corporate bonds treated here, the (X_t^λ, X_t^π)-combinations that give a perfect fit are hyperbola-like curves in the relevant range of the state variables.

[35] A value of X_t^λ above the true value gives rise to a large spread, this is counterbalanced by letting the value of the recovery risk factor also be above the true value, and vice versa in the case of joint understatement.



(a) (X_t^λ, X_t^π)-combinations that induce a perfect fit for the 7% corporate bonds with maturities 1, 5, and 10 years, respectively. The grey line shows the impact of adding a measurement error of -2 bps to the observed 7% 1-year yield.

(b) (X_t^λ, X_t^π)-combinations that induce a perfect fit for the 4% corporate bonds with maturities 1, 5, and 10 years, respectively.

Figure 4: Illustration of the identification problem inherent in the corporate bond yields simulated. The observed yields are calculated by fixing (X_t^λ, X_t^π) at (0.005,0). The parameters are fixed at the true values in the simulated Model B, See Table 4.

decomposing corporate bond yields?

The premium of a default swap is given by the formula

$$S(0,T) = \frac{E^Q\Big[\int_0^T (1-\pi_s)\lambda_s e^{-\int_0^s (r_u+\lambda_u)du} ds|\mathcal{F}_0\Big]}{E^Q\Big[\sum_{i=1}^N \delta_i e^{-\int_0^{t_i}(r_u+\lambda_u)du} + \sum_{i=1}^N \int_{t_{i-1}}^{t_i} (s-t_{i-1})\lambda_s e^{-\int_0^s (r_u+\lambda_u)du} ds|\mathcal{F}_0\Big]},$$

where T is the time to maturity, $t_1, \ldots, t_N$ are the swap premium payment dates, and $\delta_i = t_t - t_{i-1}$ the time between the ith and the (i-1)th payment dates.

From this formula it is evident that the asymmetry of the default intensity λ_t and the recovery rate π_t also exists for this group of products. However, π still only appears as the product $\pi_t\lambda_t$, which was the fundamental cause of the identification problem in the case of corporate bond data.

In Figure 5, the black curves show (X_t^λ, X_t^π)-combinations that induce a perfect fit for the 1-, 5- and 10-year CDS contracts when there is no measurement noise. The grey curve illustrate what happens if a negative measurement error of -2 bps is added to the observed 1-year CDS premium. From the figure it is seen that the situation is the same as for corporate bond yields with noise. Thus replacing corporate bond yields with observed premia on CDS contracts will not completely eliminate the fundamental identification problem. But it may serve to mitigate the problem if the measurement error noise in the CDS market is smaller due to greater liquidity.

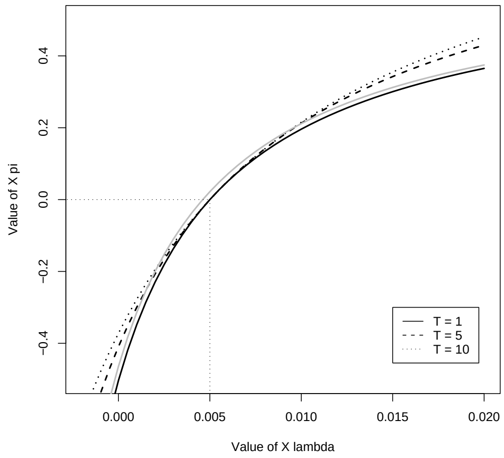


Figure 5: Illustration of the (X_t^λ, X_t^π)-combination that induces a perfect fit for individual CDS contracts with a specific observed premium. The maturities of the contracts are 1, 5, and 10 years, respectively. The observed premia are calculated by fixing (X_t^λ, X_t^π) at (0.005,0). Finally, the grey line in the graph shows the impact of adding a measurement error of -2 bps to the observed 1-year CDS premium. The parameters are fixed at the true values in the simulated Model B, See Table 4.

5 Conclusion

The purpose of this paper has been to investigate the extent to which it is possible, from observed corporate bond yields, to identify a default intensity component as well as a recovery risk component. To that end, it is argued that the appropriate and most realistic recovery assumption to use is the Recovery of Face Value assumption. This recovery assumption, in addition to being supported by empirical data, has the salient property that it actually allows for an identification of both default and recovery risk from observed corporate bond yields.

In order to exploit this in a setting that is as realistic as possible, a three-factor model containing interest rate risk, default intensity risk, and recovery rate risk is set up and simulated using a set of parameters that is chosen, with care, based on results from the empirical literature on corporate bond yields and studies of recovery rates of defaulted debt claims. Subsequently, the simulated corporate bond yields are used to perform Kalman filter estimations that reveal whether it is possible to not only obtain the true parameters used in the simulations, but to replicate the paths as well.

The results turn out to be fairly benevolent to the path and parameters of the default intensity process, and less so to those of the recovery rate process. More importantly, it turns out that the volatility of the measurement error in the observed yields plays a critical role. If this noise component is not too large, it is indeed possible to decompose yields into default and recovery risk factors with satisfactory accuracy. On the other hand, if the noise volatility is significant it is not possible to obtain a satisfactory decomposition by merely observing corporate bond yields, it is then that additional financial products must be included in the sample of observations.

Despite the mixed results, the methods used in this paper can be applied to real corporate bond data and still deliver valuable insights into the market's perception of recovery risk as well as default intensity risk.

The separation of the default intensity process λ_t and the recovery rate process π_t is essential in many settings, some of which are detailed here. Pricing of contingent claims: The premium payment leg of many credit derivatives is priced solely based on information about λ_t, i.e. only to have information about the mean loss rate $\pi_t \lambda_t$ is insufficient. Risk management: In order to manage risk effectively it is essential to not only know the distribution of default arrivals but the distribution of default severity as well. Bank regulation: In the new Basel II-rules for the internal ratings-based approach, banks are required to deliver an estimate for both the one-year probability of default and the expected loss given default. To be able to derive information about λ_t and π_t from market data could prove to be a vital input to the information derived from the banks' internal models. Future research in this area could be directed at both the theoretical development and empirical analysis of the components of credit risk in these settings, which are limited at present.

6 Appendix A

This appendix will prove the result in Proposition 2, where the process for the state variable is of the following type

$$dX_t = [\mu^0(t) + \mu^1(t)X_t]dt + \Sigma(t)dW_t,$$

while the interest rate is assumed to be a quadratic form of X_t

$$r(X_t, t) = X_t' A^r(t) X_t + B^r(t)' X_t + C^r(t).$$

The claim now goes that

$$E[e^{-\int_t^T r(X_s,s)ds} e^{X_T' \overline{A} X_T + \overline{B}' X_T + \overline{C}} [X_T' \overline{D} X_T + \overline{E}' X_T + \overline{F}] | \mathcal{F}_t] =$$

$$e^{X_t' A(t,T) X_t + B(t,T)' X_t + C(t,T)} [X_t' D(t,T) X_t + E(t,T)' X_t + F(t,T)],$$

where $A(t,T)$, $B(t,T)$, $C(t,T)$, $D(t,T)$, $E(t,T)$, and $F(t,T)$ are solutions to a set of ODEs.

First, define the following two functions

$$\begin{aligned}
\Psi_t &= e^{-\int_0^t r(X_s,s)ds} e^{X_t' A(t,T) X_t + B(t,T)' X_t + C(t,T)}, \\
\Phi_t &= e^{-\int_0^t r(X_s,s)ds} e^{X_t' A(t,T) X_t + B(t,T)' X_t + C(t,T)} [X_t' D(t,T) X_t + E(t,T)' X_t + F(t,T)].
\end{aligned}$$

Now, apply Ito's lemma to the Φ_t-function to obtain

$$\begin{aligned}
d\Phi_t &= -r(X_t, t)\Phi_t dt \\
&+ \Phi_t \Big[X_t' \frac{dA(t,T)}{dt} X_t + \frac{dB(t,T)}{dt}' X_t + \frac{dC(t,T)}{dt} \Big] dt \\
&+ \Phi_t [dX_t' A(t,T) X_t + X_t' A(t,T) dX_t] + \Phi_t B(t,T)' dX_t \\
&+ \frac{1}{2}\Phi_t [2 dX_t' A(t,T) dX_t] \\
&+ \frac{1}{2}\Phi_t [dX_t' A(t,T) X_t + X_t' A(t,T) dX_t + B(t,T)' dX_t] \\
&\times [dX_t' A(t,T) X_t + X_t' A(t,T) dX_t + B(t,T)' dX_t] \\
&+ \Psi_t \Big[X_t' \frac{dD(t,T)}{dt} X_t + \frac{dE(t,T)}{dt}' X_t + \frac{dF(t,T)}{dt} \Big] dt \\
&+ \Psi_t \Big[dX_t' D(t,T) X_t + X_t' D(t,T) dX_t + E(t,T)' dX_t \Big] \\
&+ \frac{1}{2}\Psi_t \Big[2 dX_t' D(t,T) dX_t \Big] \\
&+ 2 \times \frac{1}{2}\Psi_t [\underline{dX_t' A(t,T) X_t} + X_t' A(t,T) dX_t + B(t,T)' dX_t] \\
&\times [dX_t' D(t,T) X_t + \underline{X_t' D(t,T) dX_t} + \underline{E(t,T)' dX_t}].
\end{aligned}$$

From Leippold and Wu (2002) it follows that the five first lines can be rearranged into the four first lines of the following expression. Besides this, the underlined elements in the remaining terms have been transposed to obtain[36]

$$
\begin{aligned}
d\Phi_t &= \Phi_t X_t' \Big[\frac{dA(t,T)}{dt} - A^r(t) + \mu^1(t)'A(t,T) + A(t,T)\mu^1(t) + 2A(t,T)\Sigma(t)\Sigma(t)'A(t,T)\Big] X_t dt \\
&+ \Phi_t X_t' \Big[\frac{dB(t,T)}{dt} - B^r(t) + 2A(t,T)\mu^0(t) + \mu^1(t)'B(t,T) + 2A(t,T)\Sigma(t)\Sigma(t)'B(t,T)\Big] dt \\
&+ \Phi_t \Big[\frac{dC(t,T)}{dt} - C^r(t) + B(t,T)\mu^0(t) + trace(\Sigma(t)'A(t,T)\Sigma(t)) + \frac{1}{2}B(t,T)'\Sigma(t)\Sigma(t)'B(t,T)\Big] dt \\
&+ \Phi_t \Big[2X_t'A(t,T) + B(t,T)'\Big]\Sigma(t)dW_t \\
&+ \Psi_t \Big[X_t' \frac{dD(t,T)}{dt} X_t + \frac{dE(t,T)}{dt}' X_t + \frac{dF(t,T)}{dt}\Big] dt \\
&+ \Psi_t \Big[dX_t' D(t,T) X_t + X_t' D(t,T) dX_t + E(t,T)' dX_t\Big] \\
&+ \Psi_t \Big[dX_t' D(t,T) dX_t\Big] \\
&+ \Psi_t \Big[4X_t'A(t,T)dX_t dX_t' D(t,T) X_t + 2X_t'A(t,T)dX_t dX_t' E(t,T)\Big] \\
&+ \Psi_t \Big[2B(t,T)'dX_t dX_t' D(t,T) X_t + B(t,T)'dX_t dX_t' E(t,T)\Big].
\end{aligned}
$$

Assuming that $A(t,T)$, $B(t,T)$, and $C(t,T)$ are the solutions to the ODEs in Proposition 1, the first three lines can be eliminated. In addition, the fact that $dX_t = [\mu^0(t) + \mu^1(t)X_t]dt + \Sigma(t)dW_t$ can be applied to obtain

$$
\begin{aligned}
d\Phi_t &= \Phi_t \Big[2X_t'A(t,T)\Sigma(t) + B(t,T)'\Sigma(t)\Big] dW_t \\
&+ \Psi_t \Big[X_t' \frac{dD(t,T)}{dt} X_t + \frac{dE(t,T)}{dt}' X_t + \frac{dF(t,T)}{dt}\Big] dt \\
&+ \Psi_t \Big[\big([\mu^0(t) + \mu^1(t)X_t]dt + \Sigma(t)dW_t\big)' D(t,T) X_t + X_t' D(t,T) \big([\mu^0(t) + \mu^1(t)X_t]dt + \Sigma(t)dW_t\big)\Big] \\
&+ \Psi_t E(t,T)' \big([\mu^0(t) + \mu^1(t)X_t]dt + \Sigma(t)dW_t\big) \\
&+ \Psi_t \big([\mu^0(t) + \mu^1(t)X_t]dt + \Sigma(t)dW_t\big)' D(t,T) \big([\mu^0(t) + \mu^1(t)X_t]dt + \Sigma(t)dW_t\big) \\
&+ \Psi_t \Big[4X_t'A(t,T)\Sigma(t)\Sigma(t)'D(t,T)X_t + 2X_t'A(t,T)\Sigma(t)\Sigma(t)'E(t,T)\Big] dt \\
&+ \Psi_t \Big[2B(t,T)'\Sigma(t)\Sigma(t)'D(t,T)X_t + B(t,T)'\Sigma(t)\Sigma(t)'E(t,T)\Big] dt.
\end{aligned}
$$

[36] Transposing is allowed since the terms are mere one-dimensional real numbers.

Now, collect all square terms of X_t, all terms with a single X_t', and all terms without any X_t

$$\begin{aligned}
d\Phi_t &= \Psi_t X_t'\Big[\frac{dD(t,T)}{dt} + \mu^1(t)'D(t,T) + D(t,T)\mu^1(t) + 4A(t,T)\Sigma(t)\Sigma(t)'D(t,T)\Big]X_t dt \\
&+ \Psi_t X_t'\Big[\frac{dE(t,T)}{dt} + 2D(t,T)\mu^0(t) + \mu^1(t)'E(t,T) \\
&\qquad + 2A(t,T)\Sigma(t)\Sigma(t)'E(t,T) + 2D(t,T)\Sigma(t)\Sigma(t)'B(t,T)\Big]dt \\
&+ \Psi_t\Big[\frac{dF(t,T)}{dt} + \mu^0(t)'E(t,T) + B(t,T)'\Sigma(t)\Sigma(t)'E(t,T)\Big]dt \\
&+ \Psi_t\Big[(dW_t)'\Sigma(t)'D(t,T)\Sigma(t)dW_t\Big] \\
&+ \Phi_t\Big[2X_t'A(t,T) + B(t,T)'\Big]\Sigma(t)dW_t + \Psi_t\Big[2X_t'D(t,T) + E(t,T)'\Big]\Sigma(t)dW_t.
\end{aligned}$$

If $D(t,T)$, $E(t,T)$, and $F(t,T)$ are the solutions to the following ordinary differential equations

$$\frac{dD(t,T)}{dt} = -\mu^1(t)'D(t,T) - D(t,T)\mu^1(t) - 4A(t,T)\Sigma(t)\Sigma(t)'D(t,T) \quad \text{with} \quad D(T,T) = \overline{D},$$

$$\frac{dE(t,T)}{dt} = -2D(t,T)\mu^0(t) - \mu^1(t)'E(t,T) - 2A(t,T)\Sigma(t)\Sigma(t)'E(t,T) - 2D(t,T)\Sigma(t)\Sigma(t)'B(t,T)$$

with $E(T,T) = \overline{E}$, and finally

$$\frac{dF(t,T)}{dt} = -\mu^0(t)'E(t,T) - trace(\Sigma(t)'D(t,T)\Sigma(t)) - B(t,T)'\Sigma(t)\Sigma(t)'E(t,T) \quad \text{with} \quad F(T,T) = \overline{F},$$

and it holds that

$$\int_0^t \Big(\Phi_s[2X_s'A(s,T) + B(s,T)'] + \Psi_s[2X_s'D(s,T) + E(s,T)']\Big)\Sigma(s)dW_s$$

is a martingale for all $0 \le t \le T$, then Φ_t is a martingale, which is equivalent to

$$E\Big[e^{-\int_0^T r(X_s,s)ds} e^{X_T'A(T,T)X_T + B(T,T)'X_T + C(T,T)}[X_T'D(T,T)X_T + E(T,T)'X_T + F(T,T)]\Big|\mathcal{F}_t\Big]$$

$$= e^{-\int_0^t r(X_s,s)ds} e^{X_t'A(t,T)X_t + B(t,T)'X_t + C(t,T)}[X_t'D(t,T)X_t + E(t,T)'X_t + F(t,T)].$$

This, in turn, shows that

$$E\Big[e^{-\int_t^T r(X_s,s)ds} e^{X_T'\overline{A}X_T + \overline{B}'X_T + \overline{C}}[X_T'\overline{D}X_T + \overline{E}'X_T + \overline{F}]|\mathcal{F}_t\Big]$$

$$= e^{X_t'A(t,T)X_t + B(t,T)'X_t + C(t,T)}[X_t'D(t,T)X_t + E(t,T)'X_t + F(t,T)],$$

which is exactly what was wanted. **QED**

7 Appendix B

This appendix takes the reader through the details of the extended Kalman filter as applied in the paper. The estimation is performed in two steps. First, the extended Kalman filter is applied to estimate the interest rate process. Next, given the estimated path and parameters for the r_t-process, the Kalman filter is used to jointly estimate the paths and parameters of $X_t^{i,\lambda}$ and $X_t^{i,\pi}$ for each firm.

7.1 Step 1: The Kalman filter estimation of r_t

The interest rate is assumed to be

$$dr_t = \kappa_r(\theta_r - r_t)dt + \sigma_r dW_t^r$$

Discretizing this process an affine transition equation is obtained

$$r_t = \Phi_t^{r,0}(\psi) + \Phi_t^{r,1}(\psi)r_{t-1} + u_t^r, \quad u_t^r \sim N(0, V_t^r(\psi)),$$

where

$$\begin{aligned} \Phi_t^{r,0}(\psi) &= \theta_r(1 - e^{-\kappa_r \Delta t}), \\ \Phi_t^{r,1}(\psi) &= e^{-\kappa_r \Delta t}, \\ V_t^r(\psi) &= \frac{\sigma_r^2}{2\kappa_r}(1 - e^{-2\kappa_r \Delta t}). \end{aligned}$$

Here, Δt is the same time step size as was used in the simulation, and ψ is the set of parameters to be estimated.

Due to risk premia γ_0^r and γ_1^r it holds under the Q-measure that

$$dr_t = \Big(\kappa_r\theta_r - \gamma_0^r\sigma_r - (\kappa_r + \gamma_1^r\sigma_r)r_t\Big)dt + \sigma_r d\widetilde{W}_t^r.$$

The model-implied price of a treasury bond with maturity at T and coupon rate C paid semi-annually is

$$\begin{aligned} V_t(r_t, T; \psi) &= E_t^Q[e^{-\int_t^T r_u du}] + \sum_{i=1}^N \frac{C}{2} E_t^Q[e^{-\int_t^{t_i} r_u du}] \\ &= e^{\alpha_r(T-t;\psi)+\beta_r(T-t;\psi)r_t} + \sum_{i=1}^N \frac{C}{2} e^{\alpha_r(t_i-t;\psi)+\beta_r(t_i-t;\psi)r_t}, \end{aligned}$$

where $\alpha_r(T-t;\psi)$ and $\beta_r(T-t;\psi)$ are the well-known solutions of the ODEs in the Vasicek model.

Since the yield to maturity $y_t(T-t)$ on such a bond is the solution to the equation

$$\sum_{i=1}^{N} \frac{C}{2} e^{-y_t(T-t)t_i} + e^{-y_t(T-t)T} = V_t(r_t, T; \psi),$$

it is a non-linear function of the state variable r_t

$$y_t(T-t) = z(r_t, T-t; \psi)$$

In addition, it is assumed that there is error in the measurement of yields, so the final measurement equation is given by[37]

$$y_t^i(T-t) = z(r_t, T-t; \psi) + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_\varepsilon^2).$$

This means that the set of parameters to be estimated is given by

$$\psi = (\kappa_r, \theta_r, \sigma_r, \gamma_0^r, \gamma_1^r, \sigma_\varepsilon).$$

To apply the extended Kalman filter a linear measurement equation is needed. To this end the model-implied yields are approximated by a first order Taylor expansion around the best guess of r_t in the prediction step of the Kalman filter algorithm (see below for details). This best guess is denoted by $\widehat{r}_{t|t-1}$ whereby the approximation becomes

$$z(r_t, T-t; \psi) \approx z(\widehat{r}_{t|t-1}, T-t; \psi) + \frac{\partial z(r_t, T-t; \psi)}{\partial r_t}\Big|_{r_t = \widehat{r}_{t|t-1}} (r_t - \widehat{r}_{t|t-1}).$$

Defining

$$\begin{aligned} A_t(\psi) &\equiv z(\widehat{r}_{t|t-1}, T-t; \psi) - \frac{\partial z(r_t, T-t; \psi)}{\partial r_t}\Big|_{r_t = \widehat{r}_{t|t-1}} \widehat{r}_{t|t-1}, \\ B_t(\psi) &\equiv \frac{\partial z(r_t, T-t; \psi)}{\partial r_t}\Big|_{r_t = \widehat{r}_{t|t-1}}, \end{aligned}$$

the measurement equation can be given on an affine form as

$$y_t(T-t) = A_t(\psi) + B_t(\psi) r_t + \varepsilon_t.$$

In the following, the steps of the standard Kalman filter are described. Define the total information available at time t by

$$Y_t = (y_1, y_2, \ldots, y_t).$$

[37] The measurement error is thus assumed homoskedastic and uncorrelated across maturities and across time, just as it is in the simulation.

Now, assume that we have a best guess of the value of the state variable in the last period, $\widehat{r}_{t-1}$, and a guess for its mean square error matrix $\widehat{\Sigma}_{k-1}$. In the prediction step of the Kalman filter we get the best guess of r_t given the information set Y_{t-1} which can be shown to be

$$\widehat{r}_{t|t-1} = E^P[r_t|Y_{t-1}] = \Phi_t^{r,0}(\psi) + \Phi_t^{r,1}(\psi)\widehat{r}_{t-1}.$$

The corresponding guess for the mean square error matrix is

$$\widehat{\Sigma}_{t|t-1} = \Phi_t^{r,1}(\psi)\widehat{\Sigma}_{t-1}\Phi_t^{r,1}(\psi)' + V_t(\psi).$$

In the update step the guess is improved upon by using the additional information contained in Y_t. Here it can be shown that

$$\begin{aligned}
\widehat{r}_t &= E[r_t|Y_t] = \widehat{r}_{t|t-1} + \widehat{\Sigma}_{t|t-1}B_t(\psi)'F_t^{-1}v_t, \\
\widehat{\Sigma}_t &= \widehat{\Sigma}_{t|t-1} - \widehat{\Sigma}_{t|t-1}B_t(\psi)'F_t^{-1}B_t(\psi)\widehat{\Sigma}_{t|t-1},
\end{aligned}$$

where[38]

$$\begin{aligned}
H_t(\psi) &= \sigma_\varepsilon^2 I, \\
v_t &= y_t - E[y_t|Y_{t-1}] = y_t - z(\widehat{r}_{t|t-1};\psi), \\
F_t &= cov(v_t) = B_t(\psi)\widehat{\Sigma}_{t|t-1}B_t(\psi)' + H_t(\psi).
\end{aligned}$$

The log-likelihood function for a given vector of parameters ψ can be calculated as[39]

$$\log L(y_1,\ldots,y_N;\psi) = \sum_{t=1}^{N}\Big(-\frac{m_t}{2}\log(2\pi) - \frac{1}{2}\log|F_t| - \frac{1}{2}v_t^T F_t^{-1}v_t\Big).$$

Finally, since the interest rate process is stationary under the P-measure, the algorithm is started at the unconditional mean and variance of the state variable under this measure

$$\widehat{r}_0 = \theta_r, \quad \widehat{\Sigma}_0 = \frac{\sigma_r^2}{2\kappa_r}.$$

The optimal choice of the parameters and the corresponding most likely path of r_t is found by maximizing the value of the log-likelihood function.[40]

[38] The dimension of $H_t(\psi)$ is determined by the number of bond yields observed at time t.

[39] m_t is the number of observed yields at time t.

[40] See Duffee (1999) for a similar estimation based on the CIR model for the r_t-process.

7.2 Step 2: The Kalman filter estimation of $X_t^{i,\lambda}$ and $X_t^{i,\pi}$

Given the result of the estimation of r_t, the parameters and paths of $X_t^{i,\lambda}$ and $X_t^{i,\pi}$ need to be estimated. The structure of the default intensity as well as the recovery rate is maintained as in the simulation, i.e.

$$\begin{aligned}
\lambda_t^i &= \lambda_0^i + \lambda_r^i(r_t - \theta_r) + \lambda_1^i(X_t^{i,\lambda} - \theta_\lambda^i), \\
\pi_t^i &= \pi_0^i + \pi_r^i(r_t - \theta_r) + \pi_1^i(X_t^{i,\pi} - \theta_\pi^i).
\end{aligned}$$

From these equations it follows that it will not be possible to distinguish between the factor loading λ_1^i and the mean θ_λ^i. The same holds for π_1^i and θ_π^i in the recovery rate. This identification problem is solved by fixing the means at $\theta_\lambda^i = 0.005$ and $\theta_\pi^i = 0$ (which is identical to the values used in the simulation). Given this restriction there are 15 parameters to be estimated in the case where the idiosyncratic risk factors both carry risk premia

$$\psi^i = (\lambda_0^i, \lambda_r^i, \lambda_1^i, \kappa_\lambda^i, \sigma_\lambda^i, \gamma_0^{i,\lambda}, \gamma_1^{i,\lambda}, \pi_0^i, \pi_r^i, \pi_1^i, \kappa_\pi^i, \sigma_\pi^i, \gamma_0^{i,\pi}, \gamma_1^{i,\pi}, \sigma_\varepsilon^i).$$

This number is reduced to 11 parameters if there are no idiosyncratic risk premia

$$\psi^i = (\lambda_0^i, \lambda_r^i, \lambda_1^i, \kappa_\lambda^i, \sigma_\lambda^i, \pi_0^i, \pi_r^i, \pi_1^i, \kappa_\pi^i, \sigma_\pi^i, \sigma_\varepsilon^i).$$

The above solution of the identification problem implies that it will be the products $\widetilde{X}_t^{i,\lambda} = \lambda_1^i X_t^{i,\lambda}$ and $\widetilde{X}_t^{i,\pi} = \pi_1^i X_t^{i,\pi}$ that are going to be estimated. Under the P-measure these processes are given by

$$\begin{aligned}
d\widetilde{X}_t^{i,\lambda} &= \left(\kappa_\lambda^i \lambda_1^i \theta_\lambda^i - \kappa_\lambda^i \widetilde{X}_t^{i,\lambda}\right)dt + \lambda_1^i \sigma_\lambda^i dW_t^{i,\lambda}, \\
d\widetilde{X}_t^{i,\pi} &= \left(\kappa_\pi^i \pi_1^i \theta_\pi^i - \kappa_\pi^i \widetilde{X}_t^{i,\pi}\right)dt + \pi_1^i \sigma_\pi^i dW_t^{i,\pi}.
\end{aligned}$$

Since these are Vasicek processes the transition equations are still affine

$$\begin{aligned}
\widetilde{X}_t^{i,\lambda} &= \Phi_t^{\lambda,0}(\psi^i) + \Phi_t^{\lambda,1}(\psi^i)\widetilde{X}_{t-1}^{i,\lambda} + u_t^\lambda, \quad u_t^\lambda \sim N(0, V_t^\lambda(\psi^i)), \\
\widetilde{X}_t^{i,\pi} &= \Phi_t^{\pi,0}(\psi^i) + \Phi_t^{\pi,1}(\psi^i)\widetilde{X}_{t-1}^{i,\pi} + u_t^\pi, \quad u_t^\pi \sim N(0, V_t^\pi(\psi^i)),
\end{aligned}$$

where

$$\Phi_t^{\lambda,0}(\psi^i) = \lambda_1^i\theta_\lambda^i(1-e^{\kappa_\lambda^i\Delta t}), \quad \Phi_t^{\lambda,1}(\psi^i) = e^{-\kappa_\lambda^i\Delta t}, \quad V_t^\lambda(\psi^i) = \frac{(\lambda_1^i\sigma_\lambda^i)^2}{2\kappa_\lambda^i}(1-e^{-2\kappa_\lambda^i\Delta t}),$$
$$\Phi_t^{\pi,0}(\psi^i) = \pi_1^i\theta_\pi^i(1-e^{\kappa_\pi^i\Delta t}), \quad \Phi_t^{\pi,1}(\psi^i) = e^{-\kappa_\pi^i\Delta t}, \quad V_t^\pi(\psi^i) = \frac{(\pi_1^i\sigma_\pi^i)^2}{2\kappa_\pi^i}(1-e^{-2\kappa_\pi^i\Delta t}).$$

with Δt equal to the time step size used in the simulation.

Due to the independence between $\widetilde{X}_t^{i,\lambda}$ and $\widetilde{X}_t^{i,\pi}$ the structure on matrix form is as follows[41]

$$\Phi_t^0(\psi^i) = \begin{pmatrix} \Phi_t^{\lambda,0}(\psi^i) \\ \Phi_t^{\pi,0}(\psi^i) \end{pmatrix}, \quad \Phi_t^1(\psi^i) = \begin{pmatrix} \Phi_t^{\lambda,1}(\psi^i) & 0 \\ 0 & \Phi_t^{\pi,1}(\psi^i) \end{pmatrix}, \quad V_t(\psi^i) = \begin{pmatrix} V_t^\lambda(\psi^i) & 0 \\ 0 & V_t^\pi(\psi^i) \end{pmatrix}.$$

If risk premia are attached to these processes, under the Q-measure they become

$$\begin{aligned} d\widetilde{X}_t^{i,\lambda} &= \big(\lambda_1^i(\kappa_\lambda^i\theta_\lambda^i - \gamma_0^{i,\pi}\sigma_\lambda^i) - (\kappa_\lambda^i + \gamma_1^{i,\lambda}\sigma_\lambda^i)\widetilde{X}_t^{i,\lambda}\big)dt + \lambda_1^i\sigma_\lambda^i d\widetilde{W}_t^{i,\lambda}, \\ d\widetilde{X}_t^{i,\pi} &= \big(\pi_1^i(\kappa_\pi^i\theta_\pi^i - \gamma_0^{i,\pi}\sigma_\pi^i) - (\kappa_\pi^i + \gamma_1^{i,\pi}\sigma_\pi^i)\widetilde{X}_t^{i,\pi}\big)dt + \pi_1^i\sigma_\pi^i d\widetilde{W}_t^{i,\pi}. \end{aligned}$$

Given the above setup and the estimated path and parameters of r_t, the bond price formula from the simulation is maintained

$$\begin{aligned} V_t^i(T-t) &= E_t^Q[e^{-\int_t^T(r_u+\lambda_u^i)du}] + \sum_{i=1}^N \frac{C}{2}E_t^Q[e^{-\int_t^{t_i}(r_u+\lambda_u^i)du}] \\ &+ \int_t^T E_t^Q[\pi_s^i\lambda_s^i e^{-\int_t^s(r_u+\lambda_u^i)du}]ds \\ &+ \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \frac{C}{2}\frac{s-t_{i-1}}{t_i-t_{i-1}}E_t^Q[\pi_s^i\lambda_s^i e^{-\int_t^s(r_u+\lambda_u^i)du}]ds. \end{aligned}$$

Deriving the yield from the model-implied bond price, the measurement equation can now be written as

$$y_t^i(\tau^k) = z(\widetilde{X}_t^{i,\lambda}, \widetilde{X}_t^{i,\pi}, \tau^k; \psi^i) + \varepsilon_t^i(\tau^k), \quad \varepsilon_t^i(\tau^k) \sim N(0,\sigma_\varepsilon^2).$$

where the measurement errors are assumed to be independent and homoskedastic across time and maturities.

The first order Taylor approximation around the best guess of $(\widetilde{X}_t^{i,\lambda}, \widetilde{X}_t^{i,\pi})$ in the prediction

[41] See Appendix B in Kim (2004) for the matrix form in more general settings.

step, $(\widehat{\widetilde{X}^{i,\lambda}}_{t|t-1}, \widehat{\widetilde{X}^{i,\pi}}_{t|t-1})$, is given by

$$
\begin{aligned}
z(\widetilde{X}_t^{i,\lambda}, \widetilde{X}_t^{i,\pi}, \tau^i; \psi^i) \approx{}& z(\widehat{\widetilde{X}^{i,\lambda}}_{t|t-1}, \widehat{\widetilde{X}^{i,\pi}}_{t|t-1}, \tau^i; \psi^i) \\
&+ \frac{\partial z(\widetilde{X}_t^{i,\lambda}, \widetilde{X}_t^{i,\pi}, \tau^i; \psi^i)}{\partial \widetilde{X}_t^{i,\lambda}}\Big|_{(\widetilde{X}_t^{i,\lambda}, \widetilde{X}_t^{i,\pi}) = (\widehat{\widetilde{X}^{i,\lambda}}_{t|t-1}, \widehat{\widetilde{X}^{i,\pi}}_{t|t-1})} (\widetilde{X}_t^{i,\lambda} - \widehat{\widetilde{X}^{i,\lambda}}_{t|t-1}) \\
&+ \frac{\partial z(\widetilde{X}_t^{i,\lambda}, \widetilde{X}_t^{i,\pi}, \tau^i; \psi^i)}{\partial \widetilde{X}_t^{i,\pi}}\Big|_{(\widetilde{X}_t^{i,\lambda}, \widetilde{X}_t^{i,\pi}) = (\widehat{\widetilde{X}^{i,\lambda}}_{t|t-1}, \widehat{\widetilde{X}^{i,\pi}}_{t|t-1})} (\widetilde{X}_t^{i,\pi} - \widehat{\widetilde{X}^{i,\pi}}_{t|t-1}).
\end{aligned}
$$

This means that $B_t(\psi^i) = \begin{pmatrix} B_t^{\lambda}(\psi^i) \\ B_t^{\pi}(\psi^i) \end{pmatrix}$ should be defined by

$$
B_t^{\lambda}(\psi^i) \equiv \frac{\partial z(\widetilde{X}_t^{i,\lambda}, \widetilde{X}_t^{i,\pi}, \tau^i; \psi^i)}{\partial \widetilde{X}_t^{i,\lambda}}\Big|_{(\widetilde{X}_t^{i,\lambda}, \widetilde{X}_t^{i,\pi}) = (\widehat{\widetilde{X}^{i,\lambda}}_{t|t-1}, \widehat{\widetilde{X}^{i,\pi}}_{t|t-1})}
$$

and

$$
B_t^{\pi}(\psi^i) \equiv \frac{\partial z(\widetilde{X}_t^{i,\lambda}, \widetilde{X}_t^{i,\pi}, \tau^i; \psi^i)}{\partial \widetilde{X}_t^{i,\pi}}\Big|_{(\widetilde{X}_t^{i,\lambda}, \widetilde{X}_t^{i,\pi}) = (\widehat{\widetilde{X}^{i,\lambda}}_{t|t-1}, \widehat{\widetilde{X}^{i,\pi}}_{t|t-1})}.
$$

Finally, the algorithm is started at the unconditional mean and variance of the state variables

$$
\widehat{\widetilde{X}^{i,\lambda}}_0 = \lambda_1^i \theta_\lambda^i, \quad \widehat{\widetilde{X}^{i,\pi}}_0 = \pi_1^i \theta_\pi^i, \quad \widehat{\Sigma}_0^i = \begin{pmatrix} \frac{(\lambda_1^i \sigma_\lambda^i)^2}{2\kappa_\lambda^i} & 0 \\ 0 & \frac{(\pi_1^i \sigma_\pi^i)^2}{2\kappa_\pi^i} \end{pmatrix}.
$$

The remaining details of the Kalman filter remain the same as in the estimation of the interest rate r_t.

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ψ	λ_0	λ_r	λ_1	κ_λ	σ_λ	γ_0^λ	γ_1^λ	π_0	π_r	π_1	κ_π	σ_π	γ_0^π	γ_1^π	σ_ε
λ_0	1	0.176	-0.137	-0.392	-0.047	0.824	0.023	-0.116	-0.179	-0.496	-0.395	0.058	-0.110	0.377	0.155
λ_r		1	0.150	-0.078	-0.069	0.401	0.164	-0.040	0.208	-0.004	-0.240	0.234	0.027	0.219	0.132
λ_1			1	-0.185	-0.837	-0.168	0.100	-0.204	0.139	-0.195	0.176	0.081	-0.016	-0.096	0.044
κ_λ				1	0.291	0.022	-0.282	0.185	-0.105	0.642	0.122	0.056	0.107	-0.221	-0.038
σ_λ					1	0.106	-0.026	0.092	-0.046	0.208	-0.257	0.082	-0.022	0.160	-0.060
γ_0^λ						1	0.016	-0.064	-0.221	-0.181	-0.431	0.140	-0.073	0.345	0.151
γ_1^λ							1	0.048	-0.148	-0.111	0.091	0.006	0.058	-0.154	-0.033
π_0								1	-0.180	0.456	0.373	-0.368	0.734	-0.407	0.182
π_r									1	0.052	-0.016	0.021	0.031	0.093	-0.031
π_1										1	0.243	-0.390	0.030	-0.275	0.161
κ_π											1	-0.340	0.353	-0.953	-0.037
σ_π												1	-0.101	0.325	-0.191
γ_0^π													1	-0.389	-0.009
γ_1^π														1	0.088
σ_ε															1

Table 9: Matrix of correlation coefficients for the empirical distribution of parameter estimates in Model B with risk premia on all three risk factors and a true standard deviation of measurement error equal to 1 bp. The matrix has been calculated based on the results of 50 estimations.

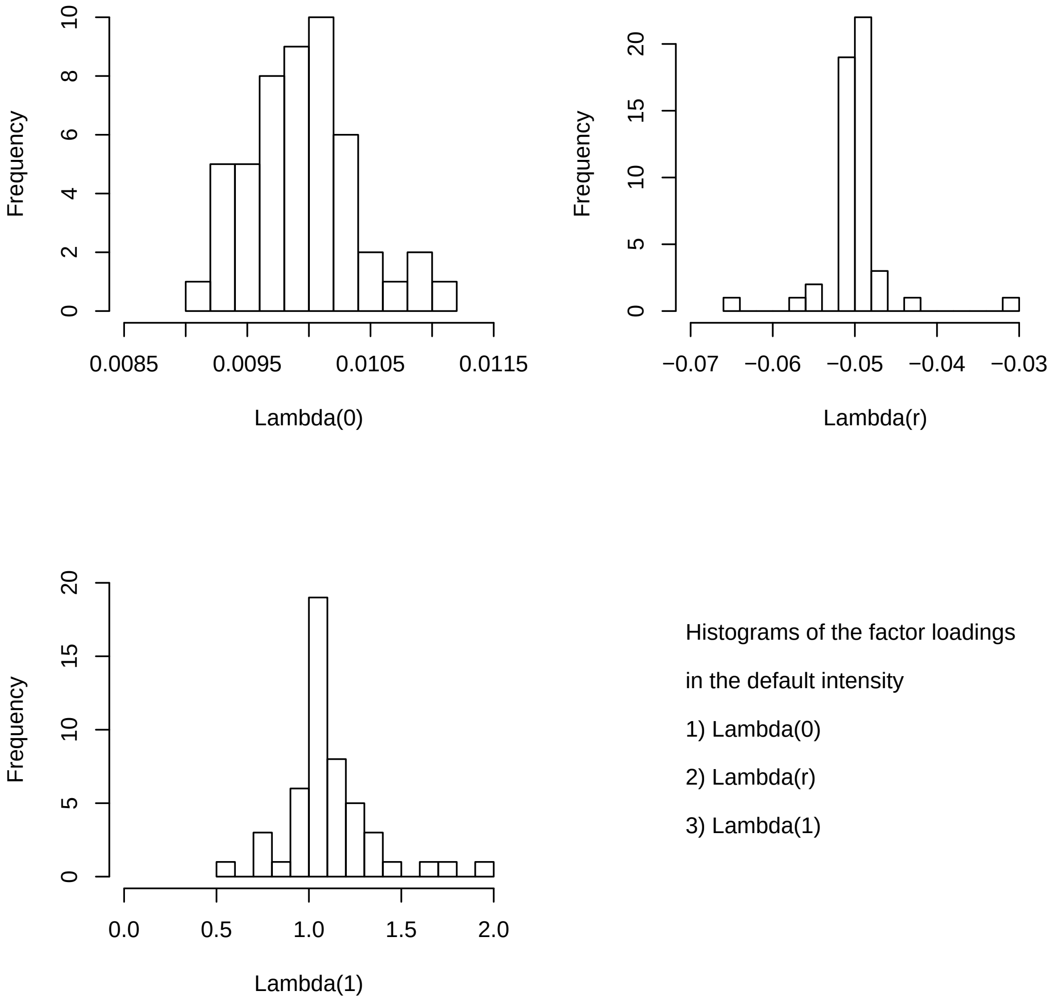

Figure 6: Histograms of the distribution of estimated parameter values for the factor loadings of the default intensity λ_t based on the simulated data set with no risk premia attached to the default and recovery risk ($\gamma_\lambda^0 = \gamma_\lambda^1 = \gamma_\pi^0 = \gamma_\pi^1 = 0$) and a standard deviation of the measurement error fixed at $\sigma_\varepsilon = 1$ bp.

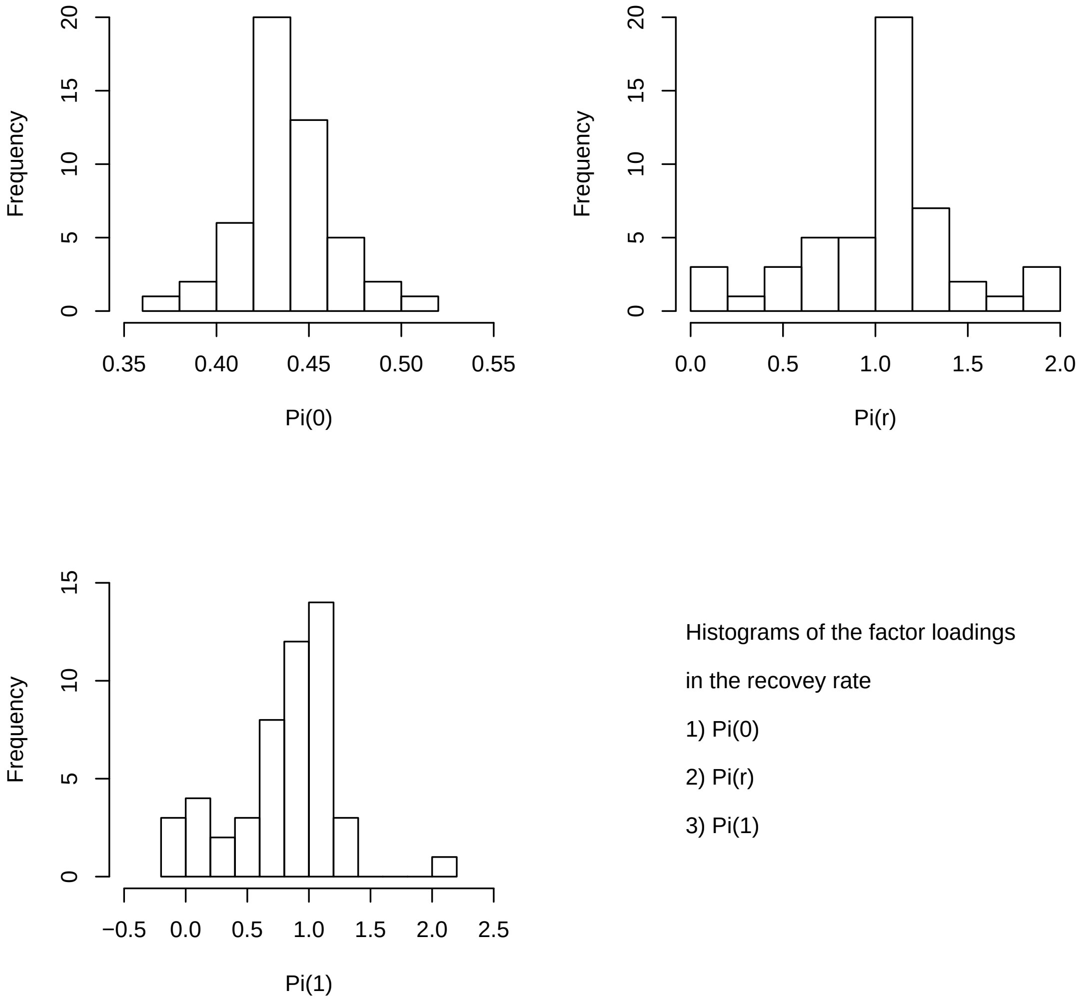

Figure 7: Histograms of the distribution of estimated parameter values for the factor loadings of the recovery rate π_t based on the simulated data set with no risk premia attached to the default and recovery risk ($\gamma_\lambda^0 = \gamma_\lambda^1 = \gamma_\pi^0 = \gamma_\pi^1 = 0$) and a standard deviation of the measurement error fixed at $\sigma_\varepsilon = 1$ bp.

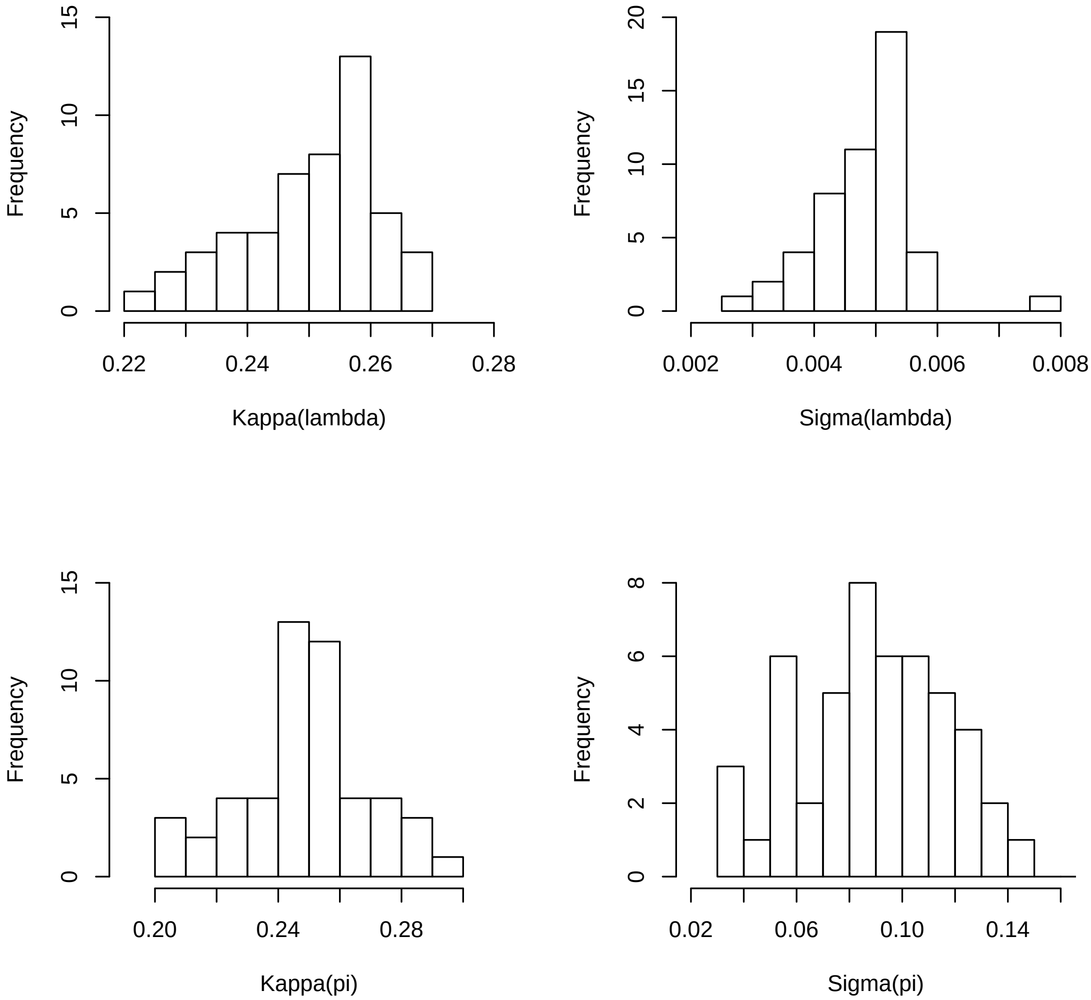

Figure 8: Histograms of the distribution of estimated parameter values for the idiosyncratic risk factors X_t^λ and X_t^π based on the simulated data set with no risk premia attached to the default and recovery risk ($\gamma_\lambda^0 = \gamma_\lambda^1 = \gamma_\pi^0 = \gamma_\pi^1 = 0$) and a standard deviation of the measurement error fixed at $\sigma_\varepsilon = 1$ bp.

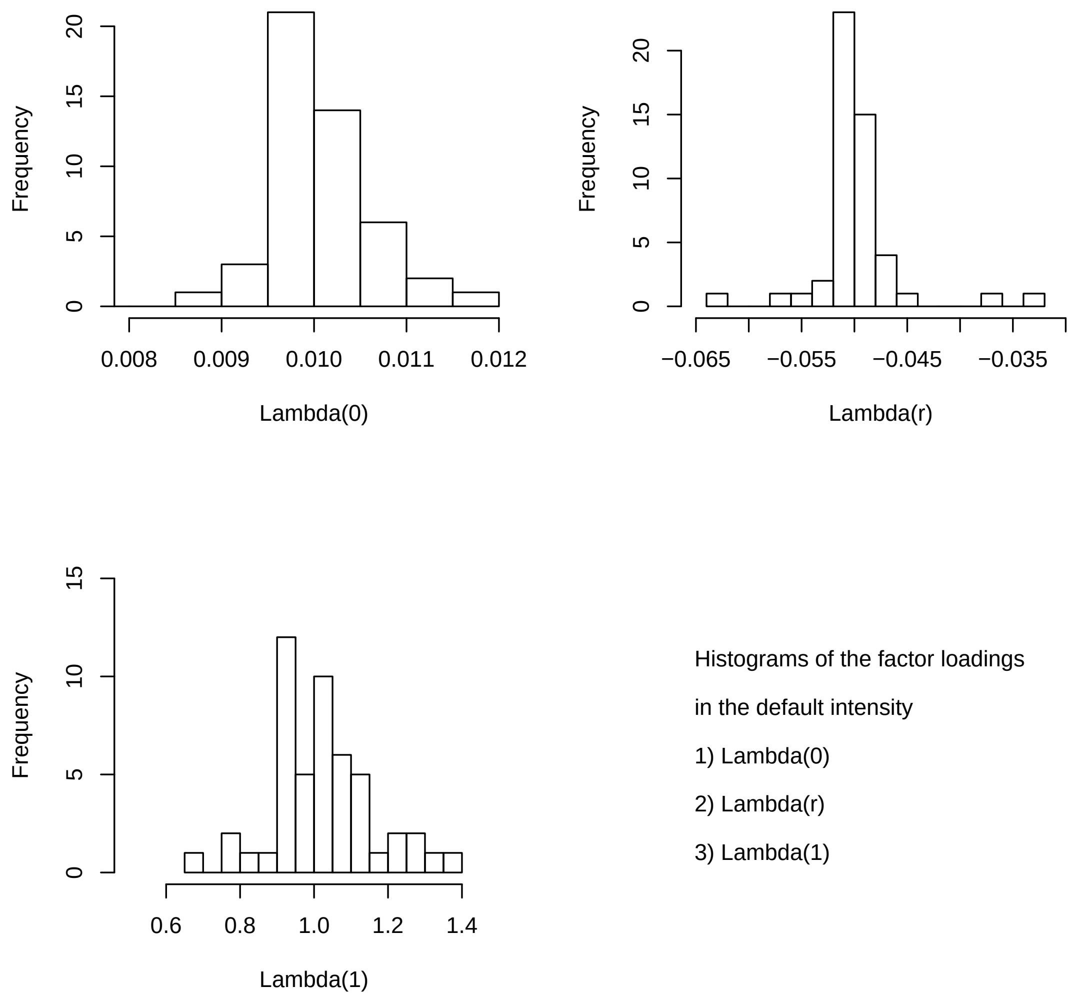

Figure 9: Histograms of the distribution of estimated parameter values for the factor loadings of the default intensity λ_t based on the simulated data set with risk premia attached to the default and recovery risk and a standard deviation of the measurement error fixed at $\sigma_\varepsilon = 1$ bp.

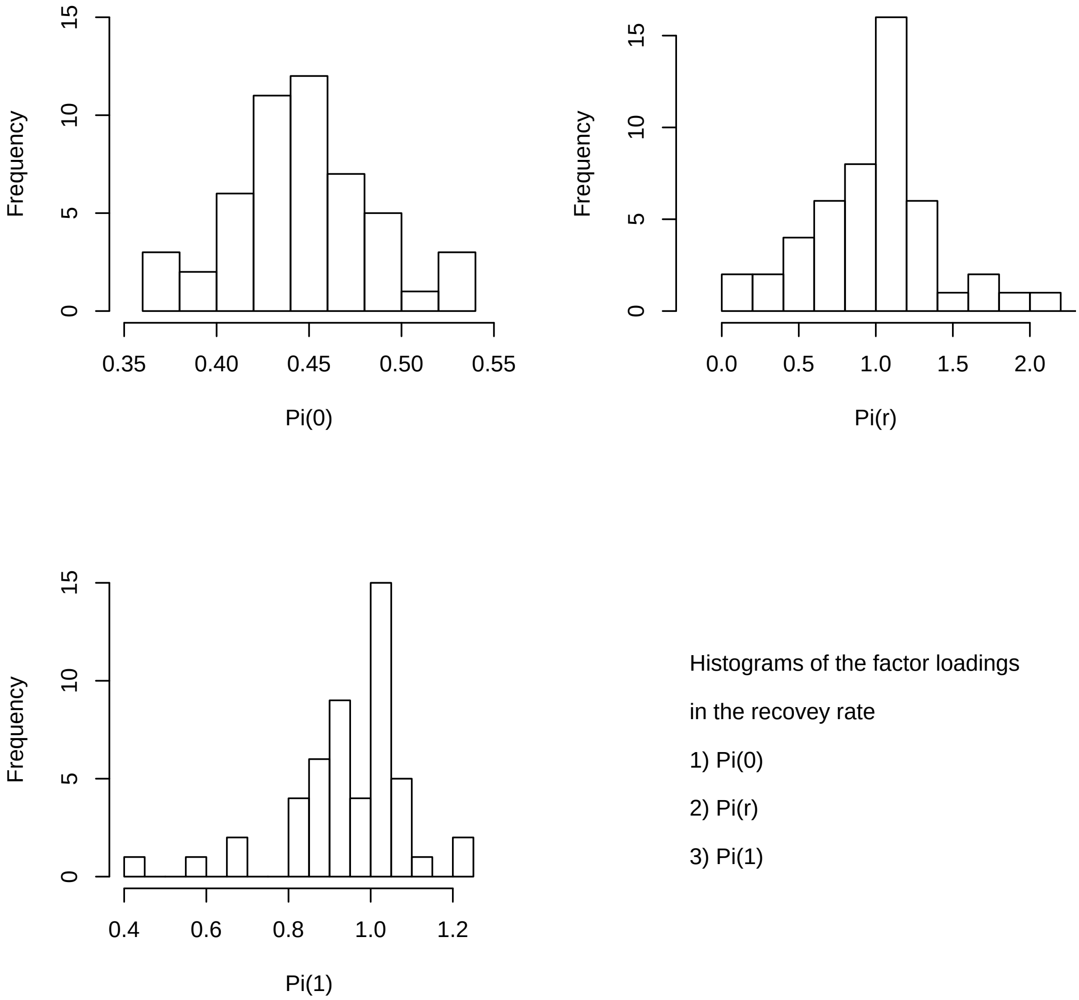

Figure 10: Histograms of the distribution of estimated parameter values for the factor loadings of the recovery rate π_t based on the simulated data set with risk premia attached to the default and recovery risk and a standard deviation of the measurement error fixed at $\sigma_\varepsilon = 1$ bp.

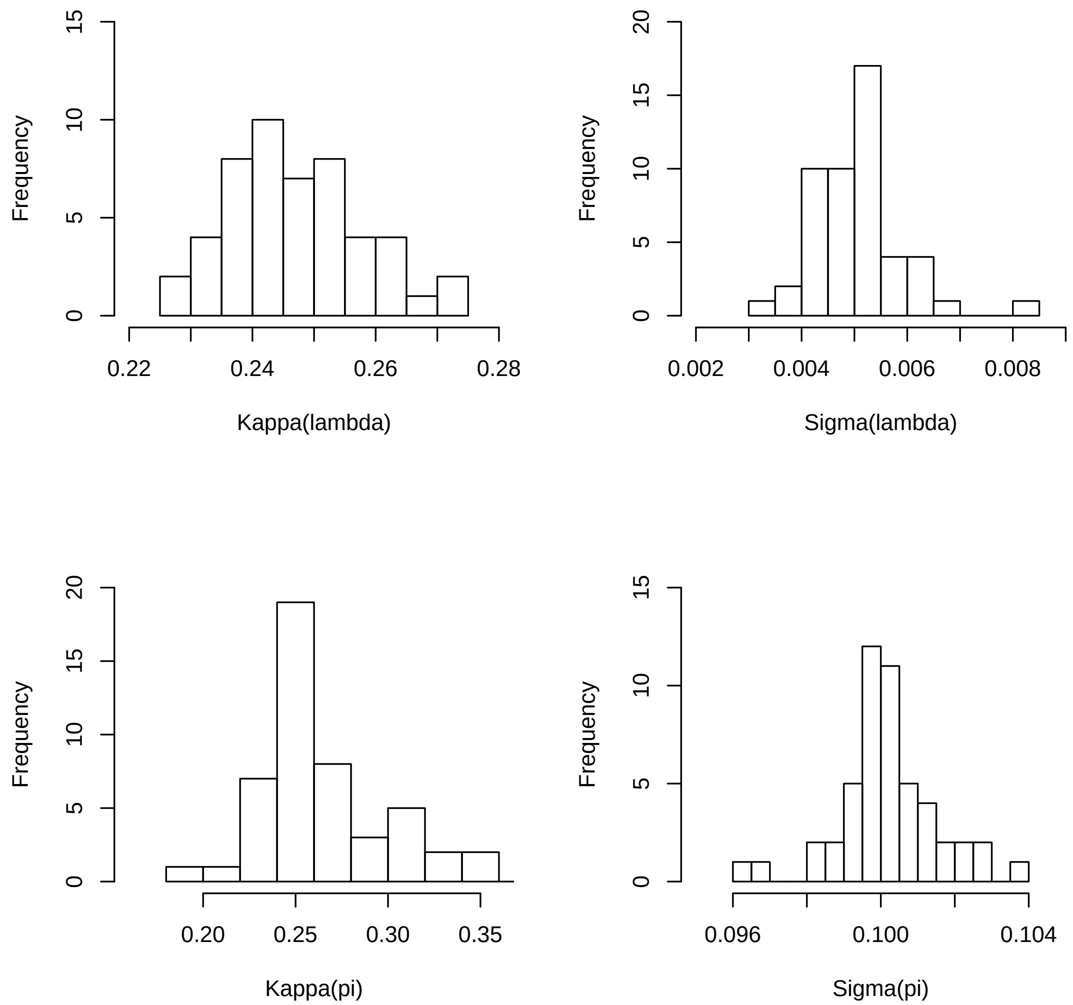

Figure 11: Histograms of the distribution of estimated parameter values for the idiosyncratic risk factors X_t^λ and X_t^π based on the simulated data set with risk premia attached to the default and recovery risk and a standard deviation of the measurement error fixed at $\sigma_\varepsilon = 1$ bp.

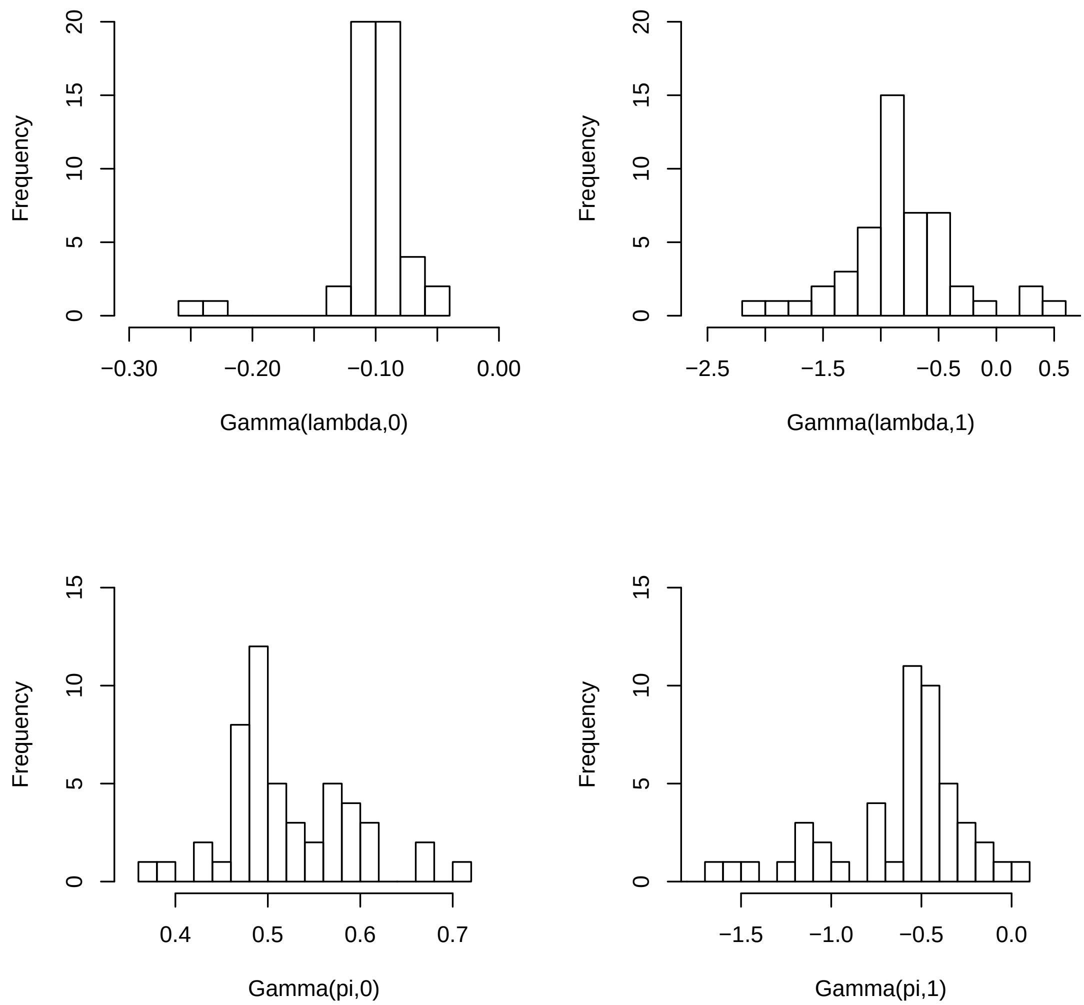

Figure 12: Histograms of the distribution of estimated parameter values for the risk premia $(\gamma_\lambda^0, \gamma_\lambda^1, \gamma_\pi^0, \gamma_\pi^1)$ of the idiosyncratic risk factors X_t^λ and X_t^π based on the simulated data set with risk premia attached to the default and recovery risk and a standard deviation of the measurement error fixed at $\sigma_\varepsilon = 1$ bp.